Notes on Number Theory and Discrete Mathematics
ISSN 1310–5132
Vol. 20, 2014, No. 4, 21–32

Generalized Euler–Seidel method for second order recurrence relations

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Abstract: We obtain identities for the generalized second order recurrence relation by using the generalized Euler–Seidel matrix with parameters x, y. As a consequence, we give some properties and generating functions of well-known special integer sequences.
Keywords: Generalized Euler–Seidel matrix, Fibonacci sequence, Lucas sequence, Pell sequence, Jacobsthal sequence, Generating function.
AMS Classification: 11B39, 11B83.

1 Introduction

Let (a_n) be a sequence. In [2], the Euler–Seidel matrix associated with this sequence is determined recursively by the formula

$$\begin{aligned} a_n^0 &= a_n \quad (n \geq 0) \\ a_n^k &= a_n^{k-1} + a_{n+1}^{k-1} \quad (n \geq 0,\ k \geq 1)\,. \end{aligned} \tag{1}$$

From relation (1), it can be seen that the first row and the first column can be transformed into each other via the well known binomial inverse pair as,

$$a_0^n = \sum_{k=0}^{n} \binom{n}{k} a_k^0, \tag{2}$$

$$a_n^0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} a_0^k. \tag{3}$$

Also any entry a_n^k can be written in terms of the initial sequence as:

$$a_n^k = \sum_{i=0}^{k} \binom{k}{i} a_{n+i}^0. \tag{4}$$

Proposition 1. *(Euler) [4] Let*

$$a(t) = \sum_{n=0}^{\infty} a_n^0 t^n$$

be the generating function of the initial sequence (a_n^0). *Then the generating function of the sequence* (a_0^n) *is*

$$\overline{a}(t) = \sum_{n=0}^{\infty} a_0^n t^n = \frac{1}{1-t} a\left(\frac{t}{1-t}\right). \tag{5}$$

Proposition 2. *(Seidel) [9] Let*

$$A(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!}$$

be the exponential generating function of the initial sequence (a_n^0). *Then the exponential generating function of the sequence* (a_0^n) *is*

$$\overline{A}(t) = \sum_{n=0}^{\infty} a_0^n \frac{t^n}{n!} = e^t A(t). \tag{6}$$

In fact, it is possible to state a more general result than (6). The following equation gives relation between exponential generating function of columns (or rows) with the exponential generating function of the initial sequence (see [2]).

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n^k \frac{u^k}{k!} \frac{t^n}{n!} = e^u A\left(t+u\right). \tag{7}$$

In [7] there are applications of Euler–Seidel matrix for hyperharmonic and $r-$Stirling numbers. Also authors introduced ”symmetric infinite matrix” and give some applications in [3].

In [5] the generalized second order recurrence sequence $\{W_n\left(a,\, b;\, p,\, q\right)\}$ is defined as for $n \geq 0$

$$W_{n+2} = pW_{n+1} - qW_n \tag{8}$$

with initial conditions

$$W_0 = a\ ,\ \ W_1 = b,$$

where $p^2 - 4q > 0$. Let the roots of the equation $t^2 - pt + q = 0$ be $\alpha = \frac{p+\sqrt{p^2-4q}}{2}$ and $\beta = \frac{p-\sqrt{p^2-4q}}{2}$. Then W_n can be written in the form

$$W_n = A\alpha^n + B\beta^n, \tag{9}$$

where $A = \frac{b-a\beta}{\alpha-\beta}$ and $B = \frac{a\alpha-b}{\alpha-\beta}$. The following generating functions of $\{W_n\}$ are given in [6, 8] as:

$$\sum_{n=0}^{\infty} W_n t^n = \frac{a + (b - pa)\, t}{1 - pt + qt^2} \tag{10}$$

and

$$\sum_{n=0}^{\infty} W_n \frac{t^n}{n!} = Ae^{\alpha t} + Be^{\beta t}. \tag{11}$$

Mező gave the generating functions of the general second-order recurrence relations in [8]. Here, we get some relation and generating functions of the general second-order recurrence relations by using generalized Euler–Seidel matrices.

The special cases of $\{W_n(a,\ b;\ p,\ q)\}$ give Fibonacci numbers F_n (Oeis A000045), Lucas numbers L_n (Oeis A000032), Pell numbers (or Silver Fibonacci numbers) P_n (Oeis A000129), Pell–Lucas numbers Q_n (Oeis A002203), Jacobsthal numbers J_n (Oeis A001045), Jacobsthal–Lucas numbers j_n (Oeis A014551), Bronze Fibonacci numbers $\mathcal{B}_n$ (Oeis A006190), Signed Fibonacci numbers $\mathcal{F}_n$ (Oeis A039834), Signed Pell numbers $\mathcal{P}_n$ (Oeis A215936).

Also we get the sequences; D_n: denominators of continued fraction convergents to $\sqrt{5}$ (Oeis A001076) and N_n: numerators of continued fraction convergents to $\sqrt{2}$ (Oeis A001333) as follows:

$$\begin{array}{ll}
W_n(0,\ 1;\ 1,\ -1) = F_n, & W_n(2,\ 1;\ 1,\ -1) = L_n, \\
W_n(0,\ 1;\ 2,\ -1) = P_n, & W_n(2,\ 2;\ 2,\ -1) = Q_n, \\
W_n(0,\ 1;\ 1,\ -2) = J_n, & W_n(2,\ 1;\ 1,\ -2) = j_n, \\
W_n(0,\ 1;\ 3,\ -1) = \mathcal{B}_n, & W_n(1,\ 1;\ -1,\ -1) = \mathcal{F}_n, \\
W_n(0,\ 1; -2,\ -1) = \mathcal{P}_n, & W_n(0,\ 1;\ 4,\ -1) = D_n, \\
W_n(1,\ 1;\ 2,\ -1) = N_n. &
\end{array}$$

2 Generalized Euler–Seidel matrices with two parameters

In this section, we consider the generalized Euler–Seidel matrix, which is given in [1] with parameters $x,\ y$. We obtain the connection between the generating functions of the initial sequence and the first column entries of the generalized Euler–Seidel matrices.

Let us consider a given sequence $(a_n)_{n\geq 0}$. Generalized Euler–Seidel matrix with parameters x and y (see [1]) corresponding to this sequence is recursively defined by the formulae

$$\begin{aligned}
a_n^0 &= a_n \quad (n \geq 0) \\
a_n^k(x,y) &= xa_n^{k-1} + ya_{n+1}^{k-1} \quad (n \geq 0,\ k \geq 1 \text{ positive integers}).
\end{aligned} \tag{12}$$

where a_n^k represents the k-th row and n-th column entry and x and y are nonzero real parameters; i.e;

$$\left(\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & a_n^{k-1} & a_{n+1}^{k-1} & \cdot & \cdot & \cdot \\
 & & & x\downarrow & \swarrow y & & & \\
\cdot & \cdot & \cdot & a_n^k & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right).$$

From now on for the sake of simplicity we represent $a_n^k(x,y)$ with a_n^k.

The following proposition gives the relation between the any entry of the matrix and the initial sequence.

Proposition 3. *[1] We have*

$$a_n^k = \sum_{i=0}^{k} \binom{k}{i} x^{k-i} y^i a_{n+i}^0. \tag{13}$$

Proof. By induction on $n+k$. □

The first row and column can be transformed into each other via the well known binomial inverse pair as follows.

Corollary 4.

$$a_0^n = x^n \sum_{i=0}^{n} \binom{n}{i} \left(\frac{y}{x}\right)^i a_i^0 \tag{14}$$

and

$$a_n^0 = \frac{1}{y^n} \sum_{i=0}^{n} \binom{n}{i} (-x)^{n-i} a_0^i. \tag{15}$$

Generating Functions. We give connections between the generating functions of the initial sequences and the first column entries.

Proposition 5. *The recurrence (12) gives the following relation:*

$$\overline{a_{x,y}}(t) = \frac{1}{1-xt} a_{x,y}\left(\frac{yt}{1-xt}\right) \tag{16}$$

where

$$\overline{a_{x,y}}(t) = \sum_{n=0}^{\infty} a_0^n t^n \quad \textit{and} \quad a_{x,y}(t) = \sum_{n=0}^{\infty} a_n^0 t^n.$$

Proof. Considering (12) we write

$$\overline{a_{x,y}}(t) = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r a_r^0 \right) t^n.$$

By changing the order of the above sums and using Newton binomial sums formula we obtain

$$\begin{aligned}\overline{a_{x,y}}(t) &= \sum_{r=0}^{\infty}\left(\frac{y}{x}\right)^{r} a_r^0 \sum_{n=0}^{\infty}\binom{n+r}{r}(xt)^{n+r} \\ &= \frac{1}{1-xt}\sum_{r=0}^{\infty} a_r^0 \left(\frac{yt}{1-xt}\right)^{r}.\end{aligned}$$

This completes the proof. □

Now we give the generalization of the equation (7).

Proposition 6. *For the a_n^k entries of the Generalized Euler–Seidel Matrices we have:*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n^k \frac{u^k}{k!}\frac{t^n}{n!} = e^{xu} A_{x,y}(t+yu)$$

where

$$A_{x,y}(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!}.$$

Proof. Using (13) we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k}\binom{k}{i} x^{k-i} y^i a_{n+i}^0\right)\frac{u^k}{k!}\frac{t^n}{n!} = \sum_{k=0}^{\infty}\sum_{i=0}^{k}\frac{x^{k-i}u^{k-i}}{(k-i)!}\sum_{n=0}^{\infty} a_{n+i}^0 \frac{t^n}{n!}\frac{(yu)^i}{i!}.$$

If we write RHS by means of Cauchy product we get:

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_n^k \frac{u^k}{k!}\frac{t^n}{n!} = \sum_{k=0}^{\infty}\frac{(xu)^k}{k!}\sum_{n=0}^{\infty}\left(a_{n+k}^0\frac{t^n}{n!}\right)\frac{(yu)^k}{k!}.$$

We can equally well write the last sum in the form $A_{x,y}(t+yu)$, which completes the proof. □

The following corollary also provides the connection between the exponential generating functions of the initial sequence and the first column entries.

Corollary 7. *[1] The following relation holds:*

$$\overline{A}_{x,y}(t) = e^{xt} A_{x,y}(yt) \tag{17}$$

where

$$\overline{A}_{x,y}(t) = \sum_{n=0}^{\infty} a_0^n \frac{t^n}{n!} \quad \textit{and} \quad A_{x,y}(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!}.$$

3 Applications of generalized Euler–Seidel matrix

In this section, we show that the generalized Euler–Seidel method is useful to obtain some properties of the generalized second order recurrence relation.

Proposition 8.

$$W_{n+2k}=\sum_{i=0}^{k}\binom{k}{i}(-q)^{k-i}p^{i}W_{n+i}. \tag{18}$$

Proof. By setting $x=-q$ and $y=p$ in (12), we obtain

$$a_n^k=-qa_n^{k-1}+pa_{n+1}^{k-1}. \tag{19}$$

For $a_n^0=W_n,\ n\geq 0$. We can write $a_n^1=W_{n+2}$. By induction on k and using equation (19), we obtain $a_n^k=W_{n+2k}$. Now considering equation (13) for $x=-q$ and $y=p$, we have

$$a_n^k=\sum_{i=0}^{k}\binom{k}{i}(-q)^{k-i}p^{i}a_{n+i}^0.$$

Then we obtain

$$W_{n+2k}=\sum_{i=0}^{k}\binom{k}{i}(-q)^{k-i}p^{i}W_{n+i}.$$

This completes the proof. □

Using (18), we get the following identities of the Fibonacci numbers F_n, Lucas numbers L_n, Pell numbers P_n, Pell–Lucas numbers Q_n, Jacobsthal numbers J_n, Jacobsthal–Lucas numbers j_n, Bronze Fibonacci numbers $\mathcal{B}_n$, Signed Fibonacci numbers $\mathcal{F}_n$, Signed Pell numbers $\mathcal{P}_n$, and also D_n and N_n numbers

$$F_{n+2k}=\textstyle\sum_{i=0}^{k}\binom{k}{i}F_{n+i}, \qquad L_{n+2k}=\sum_{i=0}^{k}\binom{k}{i}L_{n+i},$$

$$P_{n+2k}=\textstyle\sum_{i=0}^{k}\binom{k}{i}2^{i}P_{n+i}, \qquad Q_{n+2k}=\sum_{i=0}^{k}\binom{k}{i}2^{i}Q_{n+i},$$

$$J_{n+2k}=\textstyle\sum_{i=0}^{k}\binom{k}{i}2^{k-i}J_{n+i}, \qquad j_{n+2k}=\sum_{i=0}^{k}\binom{k}{i}2^{k-i}j_{n+i},$$

$$\mathcal{B}_{n+2k}=\textstyle\sum_{i=0}^{k}\binom{k}{i}3^{i}\mathcal{B}_{n+i}, \qquad \mathcal{F}_{n+2k}=\sum_{i=0}^{k}\binom{k}{i}(-1)^{i}\mathcal{F}_{n+i},$$

$$\mathcal{P}_{n+2k}=\textstyle\sum_{i=0}^{k}\binom{k}{i}(-2)^{i}\mathcal{P}_{n+i}, \qquad D_{n+2k}=\sum_{i=0}^{k}\binom{k}{i}4^{i}D_{n+i},$$

$$N_{n+2k}=\textstyle\sum_{i=0}^{k}\binom{k}{i}2^{i}N_{n+i}.$$

Corollary 9.

$$W_{2n}=\sum_{i=0}^{n}\binom{n}{i}(-q)^{n-i}p^{i}W_{i}, \tag{20}$$

$$W_{n}=\frac{1}{p^n}\sum_{i=0}^{n}\binom{n}{i}(q)^{n-i}W_{2i} \tag{21}$$

and

$$W_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}(-q)^{n-i}p^{i}W_{i+1}, \tag{22}$$

$$W_{n}=\frac{1}{p^{n-1}}\sum_{i=1}^{n}\binom{n-1}{i-1}(q)^{n-i}W_{2i-1}. \tag{23}$$

From (20), we obtain some formulas for these well-known sequences by the new method.

$$F_{2n}=\sum_{i=0}^{n}\binom{n}{i}F_i, \qquad L_{2n}=\sum_{i=0}^{n}\binom{n}{i}L_i,$$

$$P_{2n}=\sum_{i=0}^{n}\binom{n}{i}2^iP_i, \qquad Q_{2n}=\sum_{i=0}^{n}\binom{n}{i}2^iQ_i,$$

$$J_{2n}=\sum_{i=0}^{n}\binom{n}{i}2^{n-i}J_i, \qquad j_{2n}=\sum_{i=0}^{n}\binom{n}{i}2^{n-i}j_i,$$

$$\mathcal{B}_{2n}=\sum_{i=0}^{n}\binom{n}{i}3^i\mathcal{B}_i, \qquad \mathcal{F}_{2n}=\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^i\mathcal{F}_i,$$

$$\mathcal{P}_{2n}=\sum_{i=0}^{n}\binom{n}{i}\left(-2\right)^i\mathcal{P}_i, \qquad D_{2n}=\sum_{i=0}^{n}\binom{n}{i}4^iD_i,$$

$$N_{2n}=\sum_{i=0}^{n}\binom{n}{i}2^iN_i.$$

Here with help of equation (21), we have following identities:

$$F_n=\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^{n-i}F_{2i}, \qquad L_n=\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^{n-i}L_{2i},$$

$$P_n=\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^{n-i}P_{2i}, \qquad Q_n=\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^{n-i}Q_{2i},$$

$$J_n=\sum_{i=0}^{n}\binom{n}{i}\left(-2\right)^{n-i}J_{2i}, \qquad j_n=\sum_{i=0}^{n}\binom{n}{i}\left(-2\right)^{n-i}j_{2i},$$

$$\mathcal{B}_n=\frac{1}{3^n}\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^{n-i}\mathcal{B}_{2i}, \qquad \mathcal{F}_n=\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^i\mathcal{F}_{2i},$$

$$\mathcal{P}_n=\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^i\mathcal{P}_{2i}, \qquad D_n=\frac{1}{4^n}\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^{n-i}D_{2i},$$

$$N_n=\frac{1}{2^n}\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^{n-i}N_{2i}.$$

We show from (22)

$$F_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}F_{i+1}, \qquad L_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}L_{i+1},$$

$$P_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}2^iP_{i+1}, \qquad Q_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}2^iQ_{i+1},$$

$$J_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}2^{n-i}J_{i+1}, \qquad j_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}2^{n-i}j_{i+1},$$

$$\mathcal{B}_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}3^i\mathcal{B}_{i+1}, \qquad \mathcal{F}_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}\left(-1\right)^i\mathcal{F}_{i+1},$$

$$\mathcal{P}_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}\left(-2\right)^i\mathcal{P}_{i+1}, \qquad D_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}4^iD_{i+1},$$

$$N_{2n+1}=\sum_{i=0}^{n}\binom{n}{i}2^iN_{i+1}.$$

The similar results obtained from equation (23):

$F_n = \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} F_{2i-1},$ $L_n = \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} L_{2i-1},$

$P_n = \frac{1}{2^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} P_{2i-1},$ $Q_n = \frac{1}{2^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} Q_{2i-1},$

$J_n = \sum_{i=1}^{n} \binom{n-1}{i-1} (-2)^{n-i} J_{2i-1},$ $j_n = \sum_{i=1}^{n} \binom{n-1}{i-1} (-2)^{n-i} j_{2i-1},$

$\mathcal{B}_n = \frac{1}{3^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} \mathcal{B}_{2i-1},$ $\mathcal{F}_n = \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{1-i} \mathcal{F}_{2i-1},$

$\mathcal{P}_n = \frac{1}{2^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{1-i} \mathcal{P}_{2i-1},$ $D_n = \frac{1}{4^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} D_{2i-1},$

$N_n = \frac{1}{2^{n-1}} \sum_{i=1}^{n} \binom{n-1}{i-1} (-1)^{n-i} N_{2i-1}.$

4 Some results on generating functions

4.1 Results on ordinary generating functions

Proposition 10. *Generating function of the even W_n numbers is*

$$\sum_{n=0}^{\infty} W_{2n} t^n = \frac{a\,(1+qt) + (b-pa)\,pt}{(1+qt)^2 - p^2 t}. \tag{24}$$

Proof. Firstly we realize that by setting $a_n^0 = W_n$ in GES we get $a_0^n = W_{2n}$ (see Eq. (19). Here by considering (16) we have

$$\overline{a_{-q,\,p}}\,(t) = \sum_{n=0}^{\infty} W_{2n} t^n = \frac{1}{1+qt} a_{-q,\,p} \left(\frac{pt}{1+qt}\right).$$

Also we know from equation (10)

$$a_{-q,\,p}\,(t) = \sum_{n=0}^{\infty} W_n t^n = \frac{a + (b-pa)\,t}{1 - pt + qt^2}$$

which completes the proof. □

Using (24), we obtain the generating functions of the Fibonacci numbers F_n, Lucas numbers L_n, Pell numbers P_n, Pell–Lucas numbers Q_n, Jacobsthal numbers J_n, Jacobsthal–Lucas numbers j_n, Bronze Fibonacci numbers $\mathcal{B}_n$, Signed Fibonacci numbers $\mathcal{F}_n$, Signed Pell numbers $\mathcal{P}_n$, and also D_n and N_n numbers, respectively.

$\sum_{n=0}^{\infty} F_{2n} t^n = \frac{t}{1-3t+t^2},$ $\sum_{n=0}^{\infty} L_{2n} t^n = \frac{2-3t}{1-3t+t^2},$

$\sum_{n=0}^{\infty} P_{2n} t^n = \frac{2t}{1-6t+t^2},$ $\sum_{n=0}^{\infty} Q_{2n} t^n = \frac{2-6t}{1-6t+t^2},$

$\sum_{n=0}^{\infty} J_{2n}t^n = \frac{t}{1-5t+4t^2}$, $\sum_{n=0}^{\infty} j_{2n}t^n = \frac{2-5t}{1-5t+4t^2}$,

$\sum_{n=0}^{\infty} \mathcal{B}_{2n}t^n = \frac{3t}{1-11t+t^2}$, $\sum_{n=0}^{\infty} \mathcal{F}_{2n}t^n = \frac{1-3t}{1-3t+t^2}$,

$\sum_{n=0}^{\infty} \mathcal{P}_{2n}t^n = \frac{-2t}{1-6t+t^2}$, $\sum_{n=0}^{\infty} D_{2n}t^n = \frac{4t}{1-18t+t^2}$,

$\sum_{n=0}^{\infty} N_{2n}t^n = \frac{1-3t}{1-6t+t^2}$.

Proposition 11. *Generating function of the odd W_n numbers is*

$$\sum_{n=0}^{\infty} W_{2n+1}t^n = \frac{(b-pa)(1+qt)+ap}{(1+qt)^2-p^2t}. \tag{25}$$

Proof. In view of the recurrence (8) we have,

$$\sum_{n=0}^{\infty} W_{2n+1}t^n = \frac{1}{p}\left[\sum_{n=0}^{\infty} W_{2n+2}t^n + q\sum_{n=0}^{\infty} W_{2n}t^n\right].$$

Employing (24) on the right in the above equation we obtain (25). □

From (25), we get the generating functions for odd indexed of these well-known sequences.

$\sum_{n=0}^{\infty} F_{2n+1}t^n = \frac{1-t}{1-3t+t^2}$, $\sum_{n=0}^{\infty} L_{2n+1}t^n = \frac{1+t}{1-3t+t^2}$,

$\sum_{n=0}^{\infty} P_{2n+1}t^n = \frac{1-t}{1-6t+t^2}$, $\sum_{n=0}^{\infty} Q_{2n+1}t^n = \frac{2+2t}{1-6t+t^2}$,

$\sum_{n=0}^{\infty} J_{2n+1}t^n = \frac{1-2t}{1-5t+4t^2}$, $\sum_{n=0}^{\infty} j_{2n+1}t^n = \frac{1+2t}{1-5t+4t^2}$,

$\sum_{n=0}^{\infty} \mathcal{B}_{2n+1}t^n = \frac{1-t}{1-11t+t^2}$, $\sum_{n=0}^{\infty} \mathcal{F}_{2n+1}t^n = \frac{1-2t}{1-3t+t^2}$,

$\sum_{n=0}^{\infty} \mathcal{P}_{2n+1}t^n = \frac{1-t}{1-6t+t^2}$, $\sum_{n=0}^{\infty} D_{2n+1}t^n = \frac{1-t}{1-18t+t^2}$,

$\sum_{n=0}^{\infty} N_{2n+1}t^n = \frac{1+t}{1-6t+t^2}$.

4.2 Results on exponential generating functions

Proposition 12. *Exponential generating function of the W_{2n} numbers is*

$$\sum_{n=0}^{\infty} W_{2n}\frac{t^n}{n!} = Ae^{(\alpha p-q)t} + Be^{(\beta p-q)t}. \tag{26}$$

Proof. For $a_n^0 = W_n$ in GES we get $a_0^n = W_{2n}$ (see Eq. (19)). Using equation (11) we get

$$\overline{A}_{-q,p}(t) = \sum_{n=0}^{\infty} W_{2n}\frac{t^n}{n!} = e^{-qt}\left(Ae^{\alpha pt} + Be^{\beta pt}\right),$$

which completes the proof. □

From (26)

$$\sum_{n=0}^{\infty} F_{2n}\frac{t^n}{n!} = \frac{e^{\left(\frac{3+\sqrt{5}}{2}\right)t} - e^{\left(\frac{3-\sqrt{5}}{2}\right)t}}{\sqrt{5}},$$

$$\sum_{n=0}^{\infty} L_{2n}\frac{t^n}{n!} = e^{\left(\frac{3+\sqrt{5}}{2}\right)t} + e^{\left(\frac{3-\sqrt{5}}{2}\right)t},$$

$$\sum_{n=0}^{\infty} P_{2n}\frac{t^n}{n!} = \frac{e^{\left(3+2\sqrt{2}\right)t} - e^{\left(3-2\sqrt{2}\right)t}}{2\sqrt{2}},$$

$$\sum_{n=0}^{\infty} Q_{2n}\frac{t^n}{n!} = e^{\left(3+2\sqrt{2}\right)t} + e^{\left(3-2\sqrt{2}\right)t},$$

$$\sum_{n=0}^{\infty} J_{2n}\frac{t^n}{n!} = \frac{e^{4t}-e^{t}}{3},$$

$$\sum_{n=0}^{\infty} j_{2n}\frac{t^n}{n!} = e^{4t}+e^{t},$$

$$\sum_{n=0}^{\infty} \mathcal{B}_{2n}\frac{t^n}{n!} = \frac{e^{\left(\frac{11+3\sqrt{13}}{2}\right)t} - e^{\left(\frac{11-3\sqrt{13}}{2}\right)t}}{\sqrt{13}},$$

$$\sum_{n=0}^{\infty} \mathcal{F}_{2n}\frac{t^n}{n!} = \frac{\left(\sqrt{5}+3\right)e^{\left(\frac{3-\sqrt{5}}{2}\right)t} + \left(\sqrt{5}-3\right)e^{\left(\frac{3+\sqrt{5}}{2}\right)t}}{2\sqrt{5}},$$

$$\sum_{n=0}^{\infty} \mathcal{P}_{2n}\frac{t^n}{n!} = \frac{e^{\left(3-2\sqrt{2}\right)t} - e^{\left(3+2\sqrt{2}\right)t}}{2\sqrt{2}},$$

$$\sum_{n=0}^{\infty} D_{2n}\frac{t^n}{n!} = \frac{e^{\left(9+4\sqrt{5}\right)t} - e^{\left(9-4\sqrt{5}\right)t}}{2\sqrt{5}},$$

$$\sum_{n=0}^{\infty} N_{2n}\frac{t^n}{n!} = \frac{e^{\left(3-2\sqrt{2}\right)t} + e^{\left(3+2\sqrt{2}\right)t}}{2}.$$

Proposition 13. *Exponential generating function of the W_{2n+1} numbers is*

$$\sum_{n=0}^{\infty} W_{2n+1}\frac{t^n}{n!} = A\left(p-\frac{q}{\alpha}\right)e^{(\alpha p-q)t} + B\left(p-\frac{q}{\beta}\right)e^{(\beta p-q)t}. \tag{27}$$

Remark 14. For the sake of simplicity we use the following representation in the proof:

$$W_e(t) = \sum_{n=0}^{\infty} W_{2n}\frac{t^n}{n!} \quad \text{and} \quad W_o(t) = \sum_{n=0}^{\infty} W_{2n+1}\frac{t^n}{n!}.$$

Proof. From equation (8) we have

$$W_o(t) - b = pW_e(t) - pa - q\int W_o(t)\,dt.$$

This, combined with (26) to gives

$$\frac{d}{dt}W_o(t) + qW_o(t) = p\frac{d}{dt}\left\{Ae^{(\alpha p-q)t} + Be^{(\beta p-q)t}\right\}.$$

Hence we have the following differential equation:

$$W_o'(t) + qW_o(t) = Ap(\alpha p - q)\, e^{(\alpha p - q)t} + Bp(\beta p - q)\, e^{(\beta p - q)t}.$$

The solution of this linear differential equation is:

$$W_o(t) = A\left(p - \frac{q}{\alpha}\right) e^{(\alpha p - q)t} + B\left(p - \frac{q}{\beta}\right) e^{(\beta p - q)t} + Ke^{-qt}.$$

Considering $W_o(0) = b$ we calculate the constant K as

$$K = b - A\left(p - \frac{q}{\alpha}\right) - B\left(p - \frac{q}{\beta}\right) = 0.$$

Combining these results and after some rearrangement we complete the proof. □

Using (26)

$$\sum_{n=0}^{\infty} F_{2n+1}\frac{t^n}{n!} = \frac{\left(1+\sqrt{5}\right)e^{\left(\frac{3+\sqrt{5}}{2}\right)t} - \left(1-\sqrt{5}\right)e^{\left(\frac{3-\sqrt{5}}{2}\right)t}}{2\sqrt{5}},$$

$$\sum_{n=0}^{\infty} L_{2n+1}\frac{t^n}{n!} = \frac{\left(1+\sqrt{5}\right)e^{\left(\frac{3+\sqrt{5}}{2}\right)t} + \left(1-\sqrt{5}\right)e^{\left(\frac{3-\sqrt{5}}{2}\right)t}}{2},$$

$$\sum_{n=0}^{\infty} P_{2n+1}\frac{t^n}{n!} = \frac{\left(1+\sqrt{2}\right)e^{\left(3+2\sqrt{2}\right)t} - \left(1-\sqrt{2}\right)e^{\left(3-2\sqrt{2}\right)t}}{2\sqrt{2}},$$

$$\sum_{n=0}^{\infty} Q_{2n+1}\frac{t^n}{n!} = \left(1+\sqrt{2}\right)e^{\left(3+2\sqrt{2}\right)t} + \left(1-\sqrt{2}\right)e^{\left(3-2\sqrt{2}\right)t},$$

$$\sum_{n=0}^{\infty} J_{2n+1}\frac{t^n}{n!} = \frac{2e^{4t}+e^{t}}{3},$$

$$\sum_{n=0}^{\infty} j_{2n+1}\frac{t^n}{n!} = 2e^{4t} - e^{t},$$

$$\sum_{n=0}^{\infty} \mathcal{B}_{2n+1}\frac{t^n}{n!} = \frac{\left(3+\sqrt{13}\right)e^{\left(\frac{11+3\sqrt{13}}{2}\right)t} - \left(3-\sqrt{13}\right)e^{\left(\frac{11-3\sqrt{13}}{2}\right)t}}{2\sqrt{13}},$$

$$\sum_{n=0}^{\infty} \mathcal{F}_{2n+1}\frac{t^n}{n!} = \frac{\left(\sqrt{5}+1\right)e^{\left(\frac{3-\sqrt{5}}{2}\right)t} + \left(\sqrt{5}-1\right)e^{\left(\frac{3+\sqrt{5}}{2}\right)t}}{2\sqrt{5}},$$

$$\sum_{n=0}^{\infty} \mathcal{P}_{2n+1}\frac{t^n}{n!} = \frac{\left(\sqrt{2}-1\right)e^{\left(3-2\sqrt{2}\right)t} - \left(\sqrt{2}+1\right)e^{\left(3+2\sqrt{2}\right)t}}{2\sqrt{2}},$$

$$\sum_{n=0}^{\infty} D_{2n+1}\frac{t^n}{n!} = \frac{\left(2+\sqrt{5}\right)e^{\left(9+4\sqrt{5}\right)t} - \left(2-\sqrt{5}\right)e^{\left(9-4\sqrt{5}\right)t}}{2\sqrt{5}},$$

$$\sum_{n=0}^{\infty} N_{2n+1}\frac{t^n}{n!} = \frac{\left(1+\sqrt{2}\right)e^{\left(3+2\sqrt{2}\right)t} + \left(1-\sqrt{2}\right)e^{\left(3-2\sqrt{2}\right)t}}{2}.$$

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