Lie Algebra Cohomology

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1 Chain Complexes

Definition 1.1. *A chain complex (C_*, d) of R-modules is a family $\{C_n\}_{n\in\mathbb{Z}}$ of R-modules, together with R-modul maps $d_n : C_n \to C_{n-1}$ such that $d \circ d = 0$. The maps d_n are called the differentials of C.*

Ker d_n is the modul of n-cycles of C denoted $Z_n = Z_n(C)$. The image of $d_{n+1} : C_{n+1} \to C_n$ is the module of n-boundaries of C, $B_n = B_n(C)$.

Definition 1.2. *The nth homology module of C is $H_n(C) = Z_n/B_n$.*

Definition 1.3. *Let C, D be chain complexes. A morphism of complexes $u : C \to D$ is a familiy of homomorphisms $u_n : C_n \to D_n$ such that $u_{n-1}d_n = d_n u_n$.*

There is the notion of a cochain complex: A cochain complex is the same as a chain complex with the $C_n s$ replaced by $C^n s$, the differentials being maps $d^n : C^n \to C^{n+1}$ $Z^n(C) =$ Ker d^n and $B^n(C) =$ im d^{n-1}. Maps between cochain complexes are defined analogously.

Note that a chain map $f : C \to D$ induces a map $H_n(C) \to H_n(D)$:

Example 1.1. *Let $f : C \to D$ be a chain map. If $x \in$ Ker d, then $df(x) = fd(x) = 0$ and if $x \in$ im d, then $x = d(y)$ for some $y \in D$ so $f(x) = fd(y) = df(y)$ and therefore f induces a map $Z(C)/B(C) \to Z(D)/B(D)$, $[x] \mapsto [f(x)]$.*

Definition 1.4. *A chain map $f : C \to D$ is null homotopic if there are maps $s_n : C_n \to D_{n+1}$ such that $f = ds + sd$. The maps s_n are called a chain contraction of f. Two chain maps f, g from C to D are called homotopc if $f - g$ is null homotopic.*

Lemma 1.1. *If f and g are chain homotopic, then they induce the same maps in homology.*

Proof. We show that $f - g$ induces the zero map. Suppose $f - g = ds + sd$. If $x \in H_n(C)$, then $f - g(x) = ds(x) + sd(x) = ds(x)$. Hence $f - g(x)$ is a n-boundary and thus represents 0 in $H_n(D)$. □

Definition 1.5. *A chain complex C_* is exact if it is exact at every C_n, i.e $\forall n : B_n(C) = Z_n(C)$.*

2 Resolutions

Definition 2.1. *A module P is projective if given a surjection $g : B \to C$ and a map $\gamma : P \to C$ there exists a map $\beta : P \to B$ such that $\gamma = g\beta$.*

$$\begin{tikzcd} & P \arrow[dl, dotted, "\beta"'] \arrow[d, "\gamma"] \\ B \arrow[r, "g"'] & C \end{tikzcd}$$

Proposition 2.1. *An R-modul is projective iff it is a direct summand of a free R-modul.*

Definition 2.2. *Let M be an R-module. A left resolution $P \xrightarrow{\epsilon} M$ of M is a complex P with $P_i = 0$ for $i < 0$ together with a map $\epsilon : P_0 :\to M$ so that the augmented complex*

$$\cdots \xrightarrow{d} P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

is exact. It is a projective resolution if P_i is projective for all i.

Lemma 2.1. *Every R-Modul has a projective resolution.*

Proof. Let M be a R-modul. There is a free modul F_0 and an exact sequence

$$0 \longrightarrow K_1 \xrightarrow{i_1} F_0 \xrightarrow{\epsilon} M \longrightarrow 0$$

Similarly, we get a exact sequence

$$0 \longrightarrow K_2 \xrightarrow{i_2} F_1 \xrightarrow{\epsilon_1} K_1 \longrightarrow 0$$

Let $d_1 : F_1 \to F_2$ be the map $i_1\epsilon_1$. We have $\text{ im } d_1 = K_1$ and $\text{ ker } d_1 = K_2$, so the row

$$\begin{tikzcd} & & F_1 \arrow[r, "d_1"] \arrow[dr, "\epsilon_1"'] & F_0 \arrow[r, "\epsilon"] & A \arrow[r] & 0 \\ 0 \arrow[r] & K_2 \arrow[ur] & & K_1 \arrow[u, "i_1"'] & & \end{tikzcd}$$

is exact. Iterating this procedure yields a free resolution which is a projective resolution. $\square$

Theorem 2.1 (Comparison Theorem). *Let $P \xrightarrow{\epsilon} M$ be a projective resolution of M and $f' : M \to N$ a map. Then for every resolution of $Q \xrightarrow{\eta} N$ of N there is a chain map $f : P \to Q$ lifting f' in the sense that $\eta \circ f_0 = f' \circ \epsilon$. The chain map f is unique up to chain homotopy equivalence.*

$$\begin{tikzcd} \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dotted] & P_0 \arrow[r, "\epsilon"] \arrow[d, dotted] & M \arrow[r] \arrow[d, "f'"] & 0 \\ \cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r, "\eta"] & N \arrow[r] & 0 \end{tikzcd}$$

Proof. We will construct the f_n and show their uniqueness by induction on n, thinking of f_1 as f'. Inductively, suppose f_i has been constructed for $i \leq n$ so that $f_{i-1}d = df_i$. In order to construct f_{n+1} we consider the n-cycles of P and Q. If $n = -1$ we set $Z_{-1}(P) = M$ and $Z_{-1}(Q) = N$; if $n \geq 0$, the fact that $f_{n-1}d = df_n$ means that f_n induces a map f'_n from $Z_n(P)$ to $Z_n(Q)$. We have the following two diagrams with exact rows:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[d, dotted] & Z_n(P) \arrow[r] \arrow[d, "f'_n"] & 0 \\
\cdots \arrow[r] & Q_{n+1} \arrow[r] & Z_n(Q) \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r] & Z_n(P) \arrow[r] \arrow[d, "f'_n"] & P_n \arrow[r] \arrow[d, "f_n"] & P_{n-1} \arrow[d, "f_{n-1}"] \\
0 \arrow[r] & Z_n(Q) \arrow[r] & Q_n \arrow[r] & Q_{n-1}
\end{tikzcd}$$

The universal lifting property of the projective P_{n+1} yields a map f_{n+1} from P_{n+1} to Q_{n+1} so that $df_{n+1} = f'_n d = f_n d$. This finishes the inductive step and proves that a chain map $f : P \to Q$ exists.

To see uniqueness of f up to chain homotopy, suppose that $g : P \to Q$ is another lift of f' and set $h = f - g$; we will construct a chain contraction $s_n : P_n \to Q_{n+1}$ of h by induction on n. If $n < 0$, then $P_n = 0$ and thus set $s_n = 0$. If $n = 0$, note that since $\eta h_0 = \eta(f' - f') = 0$, P_0 is mapped to $Z_0(Q) = d(Q_1)$. We use the lifting property of P_0 to get a map s_0 such that $h_0 = ds_0 = ds0 + s_{-1}d$. Inductively, suppose that s_i has been constructed for $i < n$ such that $d_n s_{n-1} = h_{n-1} + s_{n-2}d_{n-1}$ and consider the map $h_n - s_{n-1}d$ from P_n to Q_n. We compute that

$$d(h_n - s_{n-1}d) = dh_n - (h_{n-1} - s_{n-2}d)d = 0$$

. Therefore, $h_n - s_{n-1}d$ lands in $Z_n(Q)$, a quotient of Q_{n+1}. The lifting property of P_n yields the desired map $s_n : P_n \to Q_{n+1}$ such that $ds_n = h_n - s_{n-1}d$.

$$\begin{tikzcd}
 & P_n \arrow[dl, dotted] \arrow[d, "h-sd"] & \\
Q_{n+1} \arrow[r, "d"] & Z_n(Q) \arrow[r] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
P_n \arrow[r, "d"] \arrow[d, "h"] & P_{n-1} \arrow[r, "d"] \arrow[dl, "s"] \arrow[d, "h"] & P_{n-2} \arrow[dl, "s"] \\
Q_n \arrow[r] & Q_{n-1} &
\end{tikzcd}$$

□

In a similar vein, we have injective resolutions:

Definition 2.3. *A modul I is injective if given an injection $f : A \to B$ and a map $\alpha : A \to I$ there exists a map $\beta : B \to I$ such that $\alpha = \beta \circ f$.*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[dl, dotted, "\beta"] \\
 & I &
\end{tikzcd}$$

Definition 2.4. *Let M be an R-module. A right resolution of M is a cochain complex I with $I^i = 0$ for $i < 0$ together with a map $: M :\to I^0$ so that the augmented complex*

$$0 \longrightarrow M \longrightarrow I^0 \xrightarrow{d} I^1 \xrightarrow{d} I^2 \longrightarrow \cdots$$

is exact. It is an injective resolution if I^i is injective for all i.

Injective is the dual concept of projective. An object P in an abelian category A is projective if and only if P is injective in A^{op}. The usual translation mechanism thus gives us for every result on projective objects the dual result for injective objects. In particular, every modul has an injective resolution and there is an analogue of the comparison theorem.

3 Derived Functors

Recall that a functor $F : A \to B$ is left exact (right exact) if for any exact sequence $0 \to A \to B \to C \to 0$ the sequence $0 \to F(A) \to F(B) \to F(C)$ $(F(A) \to F(B) \to F(C) \to 0)$ is exact.

Let A be a modul. Fix a projective resolution $P \to A$. For a right exact functor F, define $L_iF(A) = H_(i)(F(P))$. Note that $F(P_1) \to F(P_0) \to F(A) \to 0$ is exact and thus $L_0F(A) \cong F(A)$. We now show that $L_iF(A)$ does not depend on the chosen projective resolution. We show that every map $A \to A'$ induces a map $L_iF(A) \to L_i(A')$ and thus the L_iF's are functors.

Lemma 3.1. *The objects $L_iF(A)$ are well defined up to natural isomorphism. That is, if $Q \to A$ is a second projective resolution, there is an isomorphism*

$$H_iF(P) \cong H_iF(Q).$$

Proof. Let $P \to A$ and $Q \to A$ be projective resolutions of A. By the comparison theorem there is a chain map $f : P \to Q$ lifting the identity on A. As remarked above, f induces a map f_* in homology, so we get a map $f_* : H_iF(P) \to H_i(F(Q)$. Any other lift f' of id_A is homotopic to f and thus $f_* = f'_*$. Similarly, we get a chain map $g : Q \to P$ lifting the identity id_A. As gf and id_P are both chain maps lifting the identity, we have $g_*f_* = (gf)_* = id_*$. As fg and id_Q both lift id_A, we find that f_* is an isomorphism. This proves the lemma. □

Corollary 3.1. *If A is projective, then $L_iF(A) = 0$ for $i \neq 0$.*

Lemma 3.2. *If $f : A \to A'$ is any map, there is a natural map $L_iF(f) : L_iF(A') \to L_iF(A)$.*

Theorem 3.1. *Each L_iF is a additive functor.*

Proof. The identity on P lifts the identity on A and $L_iF(id_P) = Id_{H_iF(P)}$. Given maps $f' : A' \to A$ and $g' : A \to A''$ we get lifts f and g. The composite gf lifts $g'f'$ and thus $g_*f_* = (gf)_*$ as required. Additivity follows similarly. □

We can show more: if F, F' are right exact functor that are naturally isomorphic, then the left derived functors are naturally isomorphic. This hinges on the fact that for a right exact additive functor F, L_*F is a universal δ-functor.

Definition 3.1. *A covariant homological δ-functor between $\mathcal{A}$ and $\mathcal{B}$ is a collection of additive functors $T_n\mathcal{A} \to \mathcal{B}$ for $n \geq 0$ together with morphisms $\delta_nT_n(C) \to T_{n-1}(A)$ defined for each short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$. By convention, $T_n = 0$ for $n < 0$.Two conditions are imposed:*

1. *For each short exact sequence as above, there is a long exact sequence*

$$\cdots \longrightarrow T_{n+1}(C) \xrightarrow{\delta} T_n(A) \longrightarrow T_n(B) \longrightarrow T_n(C) \xrightarrow{\delta} T_{n-1}(A) \longrightarrow \cdots$$

2. *For each morphism of short exact sequences from* $0 \to A' \to B' \to C' \to 0$ *to* $0 \to A \to B \to C \to 0$, *the* δ'*s give a commutative diagram*

$$\begin{tikzcd} T_n(C') \arrow[r, "\delta"] \arrow[d] & T_{n-1}(A') \arrow[d] \\ T_n(C) \arrow[r, "\delta"] & T_{n-1}(A) \end{tikzcd}$$

Example 3.1. *Homology gives a homological* δ*-functor from the category of chain complexes of* R*-modules to the category of* R*-modules.*

Definition 3.2. *A morphism* $S \to T$ *of* δ*-functors is a system of natural transformations* $S_n \to T_n$ *that commute with* δ. *A homological* δ*-functor* T *is universal if, given any other* δ*-functor* S *and a natural transformation* $f_0 : S_0 \to T_0$ *there exists a unique morphism* $f : S \to T$ *of* δ*-functors that extends* f_0.

Theorem 3.2. *Suppose* $\mathcal{A}$ *has enough projectives and* $\mathcal{A}, \mathcal{B}$ *are abelian categories. If* $F : \mathcal{A} \to \mathcal{B}$ *is a right exact functor, then the left derived functor* L_*F *is a universal* δ*-functor.*

Using this result, we can extend our theory to left exact and contravariant functors:

Let F be a left exact functor $\mathcal{A} \to \mathcal{B}$, $\mathcal{A}, \mathcal{B}$ abelian categories. Let $\mathcal{A}$ have enough injectives. Let A be an object of $\mathcal{A}$. Then we fix an injective resolution $A \to I$ of A and let $R^iF(A) = H^i(F(I))$. As before, we get $R^0F(A) \cong F(A)$. The functor F defines a right exact functor $F^{op}\mathcal{A}^{op} \to \mathcal{B}^{op}$. Thus we get a left derived functor $L_nF^{op}(A)$ and as injective resolutions in $\mathcal{A}$ become projective in $\mathcal{A}^{op}$, we have $R^iF(A) = (L_nF^{op})^{op}(A)$ Using this, one can show that all the results on left derived functors hold for the R^iF, the right derived functors.

Definition 3.3. *For each* R*-modul* A, *the functor* $F(B) = hom_R(A, B)$ *is left exact. Its right derived functors are called the Ext groups.*

$$Ext^i_R = R^i hom_R(A, -)(B).$$

Now, let $F : \mathcal{A} \to \mathcal{B}$ be a contravariant left exact functor. Then F is covariant left exact from $\mathcal{A}^{op}$ to $\mathcal{B}$. If $\mathcal{A}$ has enough projectives, then $\mathcal{A}^{op}$ has enough injectives. Hence we can define the right derived functors $R^*F(A)$ as above.

Example 3.2. $G(A) = Hom_R(A, B)$ *is contravariant and left exact, thus we get a right derived functor* R^*G.

Moreover, we have the following theorem:

Theorem 3.3.

$$Ext^n_R(A,B) = R^n hom_R(A,-)(B) = R^n hom_R(-,B)(A).$$

Lemma 3.3. *Suppose R is a commutative ring so that $Hom_R(A,B)$ and $Ext^*_R(A,B)$ are R-modules. If $\mu : A \to A$ and $\nu : B \to B$ are multiplication by $r \in R$, so are the induced endomorphisms μ^* and ν^*.*

Proof. Pick a projective resolution $P \to A$. Multiplication by r is an R-module chain map μ' (here we need that r is central); the map $\hom(\mu', B)$ on $\mathrm{Hom}(P,B)$ is multiplication by r, because it sends $f \in \mathrm{Hom}(P_n, B)$ to $f\mu'$, which takes $p \in P_n$ to $f(rp) = rf(p)$. Hence the map μ^* on the subquotient $\mathrm{Ext}^n(A,B)$ is also multiplication by r. The argument for ν is similar. □

We return to lie algebras:

4 Lie Algebra Cohomology

Definition 4.1. *Let $\mathfrak{g}$ be a Lie algebra over ground field k. A $\mathfrak{g}$-modul is a k-vector space M together with a binary product $\mathfrak{g} \times M \to M$ such that $\forall x, y \in M, a, b \in \mathfrak{g}, \alpha \in k$*

i $a(x+y) = ax + ay$ *and* $(a+b)x = ax + bx$

ii $\alpha(ax) = (\alpha a)x = a(\alpha(x))$

iii $(ab)x = a(bx)$

A module M is a trivial $\mathfrak{g}$-modul if $xm = 0$ for all $x \in \mathfrak{g}, m \in M$. Some more Lie theorie is needed: An ideal of $\mathfrak{g}$ is a k-submodule $\mathfrak{h}$ such that $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$, that is, for all $g \in \mathfrak{g}$ and $h \in \mathfrak{h}$ we have $[g, h] \in \mathfrak{h}$. An abelian Lie algebra is one in which all brackets $[x, y] = 0$. If $\mathfrak{g}$ is a Lie algebra, let $[\mathfrak{g}, \mathfrak{g}]$ be the submodule generated by all brackets. Then $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$. The quotient $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ is abelian and denoted by $\mathfrak{g}^{ab}$.

Definition 4.2. *A Lie algebra $\mathfrak{g}$ is simple if*

i. *$\mathfrak{g}$ has no ideals except 0 and itself and*

ii. $[\mathfrak{g}, \mathfrak{g}] \neq 0$

Definition 4.3. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field of characteristic 0. Then $\mathfrak{g}$ is semisimple if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 ... \oplus \mathfrak{g}_r$ with $\mathfrak{g}_i$ simple. Every ideal of $\mathfrak{g}$ is the sum of some of these factors.*

Lemma 4.1. *If $\mathfrak{g}$ is semisimple, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Therefore $\mathfrak{g}^{ab} = 0$.*

Proof. As $\mathfrak{g}_i$ is simple $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$ and therefore $[\mathfrak{g}, \mathfrak{g}] = \sum_{i,j} [\mathfrak{g}_i, \mathfrak{g}_j] = \sum_i \mathfrak{g}_i = \mathfrak{g}$. □

Theorem 4.1. *The category $\mathfrak{g}$-mod is naturally isomorphic to the category $U\mathfrak{g} -$ mod.*

Corollary 4.1. *The category $\mathfrak{g}$-mod has enough projectives.*

We can exploit this isomorphism. Given a $\mathfrak{g}$ modul map $\mathfrak{g} \to \mathrm{Lie}(A)$ there is a unique k-algebra map $U\mathfrak{g} \to A$. The map induced by the zero map in this way is denoted by $\epsilon : U\mathfrak{g} \to k$ and called augmentation. The kernel of ϵ is denoted by $\mathfrak{J}$ and called augmentation ideal. We have $U\mathfrak{g}/\mathfrak{J} = k$.

Definition 4.4. *Let $\mathfrak{g}$ be a Lie algebra over ground field k, M a $\mathfrak{g}$-modul. The n^{th} cohomology group $H^n(\mathfrak{g}, M)$ of $\mathfrak{g}$ with coefficients in M is defined as $Ext^n_{U\mathfrak{g}}(k, M)$.*

5 First Whitehead Lemma and Weyl's Theorem

Having defined $H^n(\mathfrak{g}, M)$, the nth cohomology group of a lie algebra $\mathfrak{g}$ with coefficients in M, where M is a $\mathfrak{g}$-module, we turn to calculations. Our goal is to prove the first Whitehead lemma, stating that $H^1(\mathfrak{g}, M) = 0$ for semisimple Lie algebras $\mathfrak{g}$ over fields of characteristic 0 and finite dimensional M. Our plan of attack is to first calculate $H^1(\mathfrak{g}, M)$. The general result for these groups will give us $H^1(\mathfrak{g}, k) = 0$ for semisimple $\mathfrak{g}$ and $char(k) = 0$. Together with the general result that $H^n(\mathfrak{g}, M)$ vanishes for semisimple $\mathfrak{g}$ and $char(k) = 0$, $M \neq k$ and simple, this will imply Whiteheads first lemma.

We now turn to calculating H^1. Using the above notions, we get the exact sequence $0 \to \mathfrak{J} \to U\mathfrak{g} \to k \to 0$. Applying $\mathrm{Ext}^n_{U\mathfrak{g}}(-, M)$ yields the long exact sequence

$$0 \to \mathrm{Ext}^0_{U\mathfrak{g}}(k, M) \to \mathrm{Ext}^0_{U\mathfrak{g}}(U\mathfrak{g}, M) \to \mathrm{Ext}^0_{U\mathfrak{g}}(\mathfrak{J}, M) \to \mathrm{Ext}^1_{U\mathfrak{g}}(k, M) \to \mathrm{Ext}^1_{U\mathfrak{g}}(U\mathfrak{g}, M)$$

As $\mathrm{Ext}^0_{U\mathfrak{g}}(-, M) \cong \mathrm{Hom}_{U\mathfrak{g}}(-, M)$ and by projectivity of $U\mathfrak{g}$, one obtains the exact sequence

$$0 \to \mathrm{Hom}_{U\mathfrak{g}}(k, M) \to \mathrm{Hom}_{U\mathfrak{g}}(U\mathfrak{g}, M) \to \mathrm{Hom}_{U\mathfrak{g}}(\mathfrak{J}, M) \to H^1(\mathfrak{g}, M) \to 0$$

The map $\mathrm{Hom}_{U\mathfrak{g}}(U\mathfrak{g}, M) \to M$, $\phi \mapsto \phi(1)$ is an isomorphism of abelian groups, the category of $\mathfrak{g}$ modules is isomorphic to $U\mathfrak{g}$ modules and therefore we get

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(k, M) \to M \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M) \to H^1(\mathfrak{g}, M) \to 0$$

As k is a trivial $\mathfrak{g}$ module, $f(gx) = gf(x)$ for all $g \in \mathfrak{g}, x \in k$ and $f \in \mathrm{Hom}_{\mathfrak{g}}(k, M)$ implies the ismomorphism $\mathrm{Hom}_{\mathfrak{g}}(k, M) \cong \{m \in M : xm = 0 \forall x \in \mathfrak{g}\} =: M^{\mathfrak{g}}$. Thus we are left with calculating $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M)$. To this end, we introduce a further concept.

Definition 5.1. *If M is a $\mathfrak{g}$-module, a derivation from $\mathfrak{g}$ into M is a k-linear map $D : \mathfrak{g} \to M$ such that the Leibnitz formula holds:*

$$D([x, y]) = x(Dy) - y(Dx).$$

The set of all such derivations is denoted $Der(\mathfrak{g}, M)$. It is a submodule of $Hom_k(\mathfrak{g}, M)$

Lemma 5.1. *If M is a trivial $\mathfrak{g}$-module, $Der(\mathfrak{g}, M) = Hom_k(\mathfrak{g}^{ab}, M)$.*

Proof. If $f \in \mathrm{Der}(\mathfrak{g}, M)$, one has $[\mathfrak{g}, \mathfrak{g}] \subset \ker f$, and thus f induces a homomorphism $f' : \mathfrak{g}^{ab} \to M$. If $f \in \mathrm{Hom}_k(\mathfrak{g}^{ab}, M)$, we get a map $\mathfrak{g} \to \mathfrak{g}^{ab} \to M$. Clearly, these two maps are inverse two each other and homomorphisms. □

Example 5.1 (Inner derivations)**.** *If $m \in M$, define $D_m(x) = xm$. D_m is a derivation:*

$$D_m([x,y]) = [x,y]m = x(ym) - y(xm).$$

The D_m are called the inner derivations of $\mathfrak{g}$ into M. They form a k-submodule $Der_{Inn}(\mathfrak{g}, M)$ of $Der(\mathfrak{g}, M)$:

Example 5.2. *If $\phi : \mathfrak{J} \to M$ is a $\mathfrak{g}$-map, let $D_\phi : \mathfrak{g} \to M$ be defined by $D_\phi(x) = \phi(i(x))$, i being the inclusion $\mathfrak{g} \to U\mathfrak{g}$. As*

$$D_\phi([x,y]) = \phi(i(x)i(y) - i(y)i(x)) = x\phi(i(y) - y\phi(i(x)),$$

D_ϕ is a derivation.

Using this construction, we get the following lemma:

Lemma 5.2. *The map $\phi \mapsto D_\phi$ is a natural isomorphism of k-modules:*

$$Hom_{\mathfrak{g}}(\mathfrak{J}, M) \cong Der(\mathfrak{g}, M).$$

Proof. The formula $\phi \mapsto D_\phi$ defines a homomorphism, thus it is enough to show that it is an isomorphism. We have the product map $U\mathfrak{g} \otimes_k \mathfrak{g} \to U\mathfrak{g}\mathfrak{g} = \mathfrak{J}$. The kernel of this map is the ideal generated by all the elements of the form $u \otimes [x,y] - ux \otimes y + uy \otimes x$ with $u \in U\mathfrak{g}$, $x, y \in \mathfrak{g}$.

Given a derivation $D : \mathfrak{g} \to M$, consider the map

$$f : U\mathfrak{g} \otimes_k \mathfrak{g} \to M, f(u \otimes x) = uDx$$

We have

$$\begin{aligned} f(u \otimes [x,y] - ux \otimes y + uy \otimes x) &= uD[x,y] - uxDy + uyDx \\ &= u(xDy - yDx) - uxDy + uyDx = 0. \end{aligned}$$

Hence f induces a map $\phi : \mathfrak{J} \to M$. Moreover, for $x, y \in \mathfrak{g}, u \in U\mathfrak{g}$, $\phi(y(ux)) = \phi(yux) = yuDx = y\phi(ux)$, and therefore ϕ is a $\mathfrak{g}$-module map. Since $D_\phi(x) = \phi(i(x)) = f(1 \otimes x) = Dx$ we have lifted D to $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M)$. If h is any other map with $D_h = D$ we have $\phi(ux) = uDx = uh(x) = h(ux)$, so $\phi \mapsto D_\phi$ is indeed an isomorphism. $\square$

Theorem 5.1.

$$H^1(\mathfrak{g}, M) \cong Der(\mathfrak{g}, M)/Der_{Inn}(\mathfrak{g}, M)$$

Proof. Consider the exact sequence

$$\mathrm{Hom}_{\mathfrak{g}}(U\mathfrak{g}, M) \to \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{J}, M) \to H^1(\mathfrak{g}, M) \to 0.$$

If $\phi : \mathfrak{J} \to M$ extends to a $\mathfrak{g}$-map $U\mathfrak{g} \to M$ sending 1 to $m \in M$, then

$$D_\phi(x) = \phi(x1) = xm = D_m(x).$$

Hence D_ϕ is an inner derivation. On the other hand, let D_m be inner. Then it is induced by the image of $1 \mapsto m$ in $\mathrm{Hom}_g(\mathfrak{J}, M)$ and thus we have $H^1(\mathfrak{g}, M) \cong Der(\mathfrak{g}, M)/Der_{Inn}(\mathfrak{g}, M)$. $\square$

If M is trivial, all inner derivations are zero, hence:

Corollary 5.1. *If M is a trivial $\mathfrak{g}$-modul*

$$H^1(\mathfrak{g}, M) \cong \mathit{Der}(\mathfrak{g}, M) \cong \mathit{Hom}_k(\mathfrak{g}^{ab}, M).$$

Considering k as a trivial $\mathfrak{g}$-module yields, for semisimple and finite dimensional $\mathfrak{g}$ and $char(k) = 0$:

Corollary 5.2. *If $\mathfrak{g}$ is finite-dimensional and semisimple and $char(k) = 0$ then $H^1(\mathfrak{g}, k) = 0$.*

Recall that a module M is simple if it has no proper direct sum decomposition.

Theorem 5.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0. If M is a simple $\mathfrak{g}$-module, $M \neq k$, then*

$$\forall i: \quad H^i(\mathfrak{g}, M) = 0.$$

Proof. Let C be the center of $U\mathfrak{g}$. Then $H^*(\mathfrak{g}, M) = \mathrm{Ext}^*_{U\mathfrak{g}}(k, M)$ is a C-module where the multiplication by $c \in C$ is induced by the multiplication $c : k \to k$ and $c : M \to M$. Since the Casimir element c_M acts by 0 on k (as $c_M \in \mathfrak{J}$) and by an invertible scalar r, we get $0 = r$ on $H^*(\mathfrak{g}, M)$ and thus the latter modules have to be 0. □

Corollary 5.3 (Whitehead's first lemma). *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0. If M is a finite dimensional $\mathfrak{g}$-module, then $H^1(\mathfrak{g}, M) = 0$.*

Proof. We proceed by induction on dim M. For dim $M = 0$ there is nothing to show. Assume the corollary has been shown for some dim $M = n$. Let M have dimension greater or equal than n. If M is simple, then either $M = k$ or $M \neq k$. In any case, $H^1(\mathfrak{g}, M) = 0$. Otherwise, M contains a proper submodule L. By induction, $H^1(\mathfrak{g}, L) = H^1(\mathfrak{g}, M/L) = 0$ and from the exact sequence $0 \to L \to M \to M/L \to 0$ it follows the cohomology exact sequence

$$\cdots \to H^1(\mathfrak{g}, L) \to H^1(\mathfrak{g}, M) \to H^1(\mathfrak{g}, M/L) \to \cdots,$$

whence $H^1(\mathfrak{g}, M) = 0$. □

Definition 5.2. *An extension χ of A by B is an exact sequence $0 \to B \to X \to A \to 0$. Two extensions χ, χ' are equivalent if there is a commutative diagram*

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "\cong"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & X' \arrow[r] & A \arrow[r] & 0. \end{tikzcd}$$

An extension is split if it is equivalent to $0 \to B \to A \oplus B \to A \to 0$.

Lemma 5.3. *If $Ext^1(A, B) = 0$ then every extension of A by B is split.*

Proof. Given an extension $0 \to B \to X \to A \to 0$, applying $\mathrm{Ext}^*(A, -)$ yields the exact sequence

$$\mathrm{Hom}(A, X) \to \mathrm{Hom}(A, A) \to \mathrm{Ext}^1(A, B).$$

Hence, if $\mathrm{Ext}^1(A, B) = 0$, the identity id_A lifts to a map $\sigma : A \to X$. As σ is a section of $X \to A$, the extension is split. $\square$

Theorem 5.3 (**Weyl's Theorem**). *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field of characteristic 0. Then every finite dimensional $\mathfrak{g}$-module M is completely reducible, that is, is a direct sum of simple $\mathfrak{g}$-modules.*

Proof. Suppose that M is not the direct sum of simple modules. As $\dim(M)$ is finite, M contains a submodule M_1 minimal with respect to this property. Clearly, M_1 is not simple, so it contains a proper submodule M_0. By induction, both M_0 and $M_2 = M_1/M_0$ are direct sums of simple $\mathfrak{g}$-modules yet M_1 is not, so the extension M_2 by M_0 is not split. Thus $\mathrm{Ext}^1(M_2, M_0) \neq 0$. But

$$\mathrm{Ext}^1_{U\mathfrak{g}}(M_2, M_0) \cong H^1(\mathfrak{g}, \mathrm{Hom}_k(M_2, M_0))$$

and this contradicts Whitehead's first lemma. ($\mathrm{Hom}_k(M_2, M_0)$ is a $\mathfrak{g}$-module via $(xf)(m) = xf(m) - f(xm)$.) $\square$