6-14-2008

The Fundamental Theorem of Arithmetic

- The **Fundamental Theorem of Arithmetic** says that every integer greater than 1 can be factored uniquely into a product of primes.
- **Euclid's lemma** says that if a prime divides a product of two numbers, it must divide at least one of the numbers.
- The **least common multiple** $[a, b]$ of nonzero integers a and b is the smallest positive integer divisible by both a and b.

Theorem. **(Fundamental Theorem of Arithmetic)** Every integer greater than 1 can be written in the form

$$p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$$

where $n_i \geq 0$ and the p_i's are distinct primes. The factorization is unique, except possibly for the order of the factors.

Example.

$$4312 = 2 \cdot 2156 = 2 \cdot 2 \cdot 1078 = 2 \cdot 2 \cdot 2 \cdot 539 = 2 \cdot 2 \cdot 2 \cdot 7 \cdot 77 = 2 \cdot 2 \cdot 2 \cdot 7 \cdot 7 \cdot 11.$$

That is,

$$4312 = 2^3 \cdot 7^2 \cdot 11. \quad \square$$

I need a couple of lemmas in order to prove the uniqueness part of the Fundamental Theorem. In fact, these lemmas are useful in their own right.

Lemma. If $m \mid pq$ and $(m, p) = 1$, then $m \mid q$.

Proof. Write

$$1 = (m, p) = am + bp \quad \text{for some } a, b \in \mathbb{Z}.$$

Then

$$q = amq + bpq.$$

Now $m \mid amq$ and $m \mid bpq$ (since $m \mid pq$), so $m \mid (amq + bpq) = q$. $\square$

Lemma. If p is prime and $p \mid a_1 a_2 \cdots a_n$, then $p \mid a_i$ for some i.

For $n = 2$, the result says that if p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$. This is often called **Euclid's lemma**.

Proof. Do the case $n = 2$ first. Suppose $p \mid a_1 a_2$, and suppose $p \not| a_1$. I must show $p \mid a_2$.

$(p, a_1) \mid p$, and p is prime, so $(p, a_1) = 1$ or $(p, a_1) = p$. If $(p, a_1) = p$, then $p = (p, a_1) \mid a_1$, which contradicts $p \not| a_1$. Therefore, $(p, a_1) = 1$. By the preceding lemma, $p \mid a_2$. This establishes the result for $n = 2$.

Assume $n > 2$, and assume the result is true when p divides a product of with less than n factors. Suppose that $p \mid a_1 a_2 \cdots a_n$. Grouping the terms, I have

$$p \mid (a_1 a_2 \cdots a_{n-1}) a_n.$$

By the case $n = 2$, either $p \mid a_1 a_2 \cdots a_{n-1}$ or $p \mid a_n$. If $p \mid a_n$, I'm done. Otherwise, if $p \mid a_1 a_2 \cdots a_{n-1}$, then p divides one of a_1, a_2, ..., a_{n-1}, by induction. In either case, I've shown that p divides one of the a_i's, which completes the induction step and the proof. □

Proof. (**Fundamental Theorem of Arithmetic**) First, I'll use induction to show that every integer greater than 1 can be expressed as a product of primes.

$n = 2$ is prime, so the result is true for $n = 2$.

Suppose $n > 2$, and assume every number less than n can be factored into a product of primes. If n is prime, I'm done. Otherwise, n is composite, so I can factor n as $n = ab$, where $1 < a, b < n$. By induction, a and b can be factored into primes. Then $n = ab$ shows that n can, too.

Now I'll prove the uniqueness part of the Fundamental Theorem.

Suppose that

$$p_1^{m_1} \cdots p_j^{m_j} = q_1^{n_1} \cdots q_k^{n_k}.$$

Here the p's are distinct primes, the q's are distinct primes, and all the exponents are greater than or equal to 1. I want to show that $j = k$, and that each $p_a^{m_a}$ is $q_b^{n_b}$ for some b — that is, $p_a = q_b$ and $m_a = n_b$.

Look at p_1. It divides the left side, so it divides the right side. By the last lemma, $p_1 \mid q_i^{n_i}$ for some i. But $q_i^{n_i}$ is $q_i \cdots q_i$ (n_i times), so again by the last lemma, $p_1 \mid q_i$. Since p_1 and q_i are prime, $p_1 = q_i$.

To avoid a mess, renumber the q's so q_i becomes q_1 and vice versa. Thus, $p_1 = q_1$, and the equation reads

$$p_1^{m_1} \cdots p_j^{m_j} = p_1^{n_1} \cdots q_k^{n_k}.$$

If $m_1 > n_1$, cancel $p_1^{n_1}$ from both sides, leaving

$$p_1^{m_1 - n_1} \cdots p_j^{m_j} = q_2^{n_2} \cdots q_k^{n_k}.$$

This is impossible, since now p_1 divides the left side, but not the right.

For the same reason $m_1 < n_1$ is impossible.

It follows that $m_1 = n_1$. So I can cancel the p_1's off both sides, leaving

$$p_2^{m_2} \cdots p_j^{m_j} = q_2^{n_2} \cdots q_k^{n_k}.$$

Keep going. At each stage, I pair up a power of a p with a power of a q, and the preceding argument shows the powers are equal. I can't wind up with any primes left over at the end, or else I'd have a product of primes equal to 1. So everything must have paired up, and the original factorizations were the same (except possibly for the order of the factors). □

Example. The **least common multiple** of nonzero integers a and b is the smallest positive integer divisible by both a and b. The least common multiple of a and b is denoted $[a, b]$.

For example,

$$[6, 4] = 12, \quad [33, 15] = 165.$$

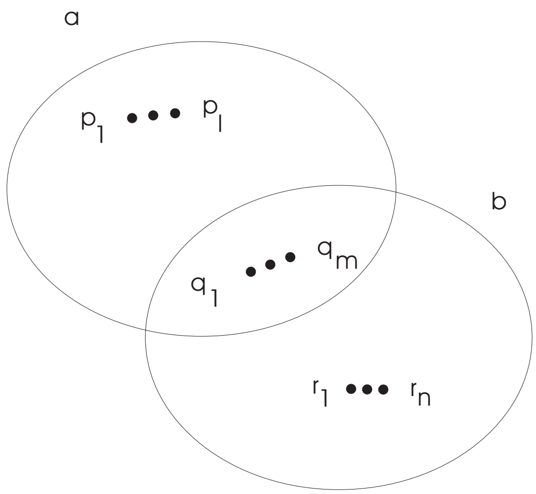
Here's an interesting fact that is easy to derive from the Fundamental Theorem:

$$[a, b](a, b) = ab.$$

Factor a and b in products of primes, but write out all the powers (e.g. write 2^3 as $2 \cdot 2 \cdot 2$):

$$a = p_1 \cdots p_l q_1 \cdots q_m, \quad b = q_1 \cdots q_m r_1 \cdots r_n.$$

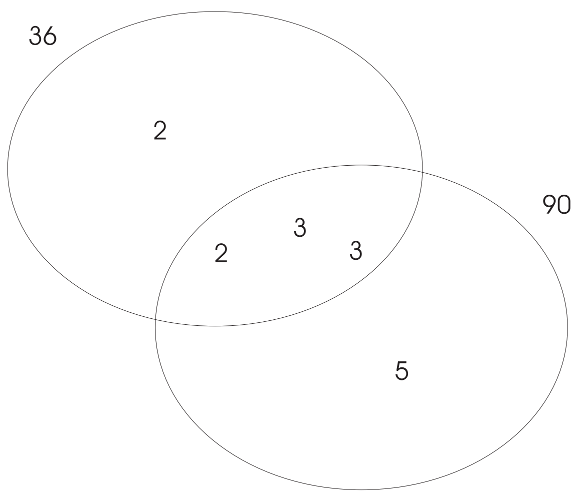
Here the q's are the primes a and b have in common, and the p's and r don't overlap. Picture:



From the picture,

$$(a, b) = q_1 \cdots q_m, \quad [a, b] = p_1 \cdots p_l q_1 \cdots q_m r_1 \cdots r_n, \quad ab = p_1 \cdots p_l q_1^2 \cdots q_m^2 r_1 \cdots r_n.$$

Thus, $[a, b](a, b) = ab$.
Here's how this result looks for 36 and 90:



$(36, 90) = 18$, $[36, 90] = 180$, and $36 \cdot 90 = 32400 = 18 \cdot 180$. □