Kepler's Laws of Planetary Motion and Newton's Law of Universal Gravitation

Abstract

These notes were written with those students in mind having taken (or are taking) AP Calculus and AP Physics. Newton's law of universal gravitation is introduced in pretty much the same way as it is in AP Physics, except that the ensuing discussion relies heavily on differential and integral calculus. Through this treatment the equation of motion is obtained for an orbiting body and the three possible trajectories (elliptical, parabolic, and hyperbolic) are classified in terms of the relevant constants. At the end is a collection of guided exercises designed to deepen the reader's understanding of the interplay between between the mathematics and the physics, as well as to provide a useful review of analytical geometry.

Equations of Planetary Motion

In this short discussion I would like to show how **Newton's law of universal gravitation** can be applied to deriving **Keplar's laws of planetary motion**. Newton's law of gravitation says that the force on a planet of mass m exerted by another planet (or star) of mass M is given by the familiar inverse-square law, and has magnitude

$$F = \frac{GMm}{r^2},$$

where r is the distance separating the two masses (measured from their cen-

Figure 1: Heliocentric diagram

ters). In the above, G is the universal gravitational constant, whose measured value is approximately

$$G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2.$$

As depicted in the above figure, the sun is at the origin (the "heliocentric" point of view), and $\overrightarrow{R}$ is the position vector of the orbiting planet. The vector $\overrightarrow{r}$ is the unit vector in the direction of $\overrightarrow{R}$. Since $\overrightarrow{r} \cdot \overrightarrow{r} = 1$, the product rule for differentiation shows that $\overrightarrow{r} \cdot \left(\frac{d}{dt} \overrightarrow{r} \right) = 0$; therefore if $\overrightarrow{s}$ is the unit vector in the dirction of $\frac{d}{dt} \overrightarrow{r}$, it follows that $\overrightarrow{r} \cdot \overrightarrow{s} = 0$, as well. All of this is depicted in Figure 1 above. In fact, from this picture, we see that $\overrightarrow{r}$ and $\overrightarrow{s}$ are given explicitly as

$$\overrightarrow{r} = (\cos\theta, \sin\theta), \quad \overrightarrow{s} = (-\sin\theta, \cos\theta),$$

from which it follows that

$$\frac{d}{dt}\overrightarrow{r} = \frac{d\theta}{dt}\overrightarrow{s} \quad \text{and that} \quad \frac{d}{dt}\overrightarrow{s} = -\frac{d\theta}{dt}\overrightarrow{r}.$$

The vector equation dictating the motion of of the orbiting planet is

$$-\left(\frac{GMm}{r^2}\right)\overrightarrow{r} = m\frac{d^2}{dt^2}\overrightarrow{R} \tag{1}$$

since the force on the planet is directed back towards the sun.

In order to compute $\frac{d^2}{dt^2}\overrightarrow{R}$ explicitly, note first that the velocity vector is given by

$$\overrightarrow{v} = \frac{d}{dt}\overrightarrow{R} = \frac{d}{dt}\left(r \cdot \overrightarrow{r}\right) = \frac{dr}{dt}\overrightarrow{r} + r\left(\frac{d}{dt}\overrightarrow{r}\right) = \frac{dr}{dt}\overrightarrow{r} + r\frac{d\theta}{dt}\overrightarrow{s} \tag{2}$$

The acceleration is the time derivative of Equation (2):

$$\begin{aligned}
\overrightarrow{a} &= \frac{d}{dt}\overrightarrow{v} = \frac{d}{dt}\left(\frac{dr}{dt}\overrightarrow{r} + r\frac{d\theta}{dt}\overrightarrow{s}\right) \\
&= \frac{d^2r}{dt^2}\overrightarrow{r} + 2\frac{dr}{dt}\frac{d\theta}{dt}\overrightarrow{s} + r\frac{d^2\theta}{dt^2}\overrightarrow{s} - r\left(\frac{d\theta}{dt}\right)^2\overrightarrow{r} \\
&= \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\overrightarrow{r} + \left(2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right)\overrightarrow{s}
\end{aligned}$$

However, from equation (1) we see that the acceleration vector is given by

$$\overrightarrow{a} = -\frac{GM}{r^2}\overrightarrow{r}$$

from which we conclude that

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{GM}{r^2} \tag{3}$$

and

$$2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0. \tag{4}$$

Equations (3) and (4) will be put to good use momentarily.

Keplar's Second Law

Keplar's second law says simply this: the area swept out by the planet's orbit between times t_1 and t_2, versus the area of the planet's orbit swept out between times t_3 and t_4 are equal provided that $t_2 - t_1 = t_4 - t_3$. In other words, equal areas are swept out in equal times. Referring to Figure 2, and recalling how to compute areas in polar coordinates, we get
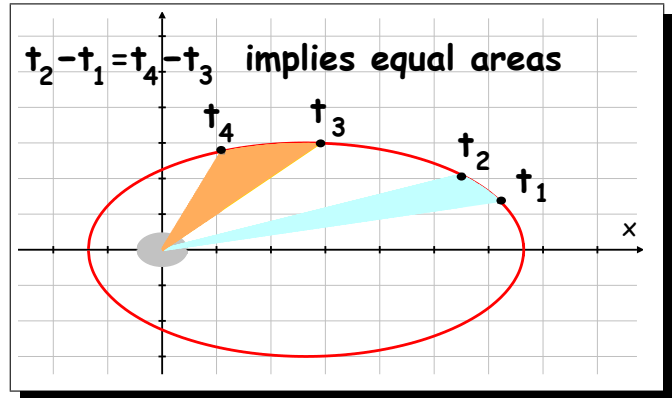

Figure 2: Keplar's Second Law

$$\text{Area swept between times } t_1 \text{ and } t_2 = \frac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta = \frac{1}{2} \int_{t_1}^{t_2} r^2 \frac{d\theta}{dt} \, dt.$$

Note, however, that

$$\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 2r \frac{dr}{dt}\frac{d\theta}{dt} + r^2 \frac{d^2\theta}{dt} = r \cdot 0 = 0,$$

where we have used Equation (4), above. The upshot is that the integrand $r^2 \dfrac{d\theta}{dt}$ is a constant—call it L—from which we conclude that

$$\text{Area swept between times } t_1 \text{ and } t_2 = \frac{1}{2} \int_{t_1}^{t_2} r^2 \frac{d\theta}{dt} \, dt = \frac{1}{2}L(t_2 - t_1).$$

Therefore, if $t_4 - t_3 = t_2 - t_1$ the areas will be the same! This proves Kepler's Second Law.

Keplar's First Law

Keplar's first law says that planets follow an eliptical orbit with the Sun at one of the foci. This is a bit trickier to prove than the second law, but we can proceed as follows. Recall from the above discussion that the quantity

$$L = r^2 \frac{d\theta}{dt}$$

is a constant; define the new constant

$$P = \frac{L^2}{GM},$$

and define the (dimensionless) variable

$$u = \frac{P}{r}.$$

This says that

$$\frac{GM}{r^2} = \frac{L^2 u^2}{P^3}$$

and that

$$\frac{d\theta}{dt} = \frac{L}{r^2} = \frac{Lu^2}{P^2}.$$

We have, using the Chain Rule, that

$$\frac{dr}{dt} = \frac{d}{d\theta}\left(\frac{P}{u}\right)\frac{d\theta}{dt} = -\frac{P}{u^2}\frac{d\theta}{dt}\frac{du}{d\theta} = -\frac{L}{P}\frac{du}{d\theta}.$$

Differentiate again and obtain

$$\frac{d^2 r}{dt^2} = \frac{d}{d\theta}\left(\frac{dr}{dt}\right)\frac{d\theta}{dt} = -\frac{L}{P}\frac{d^2 u}{d\theta^2}\frac{d\theta}{dt} = -\frac{L^2 u^2}{P^3}\frac{d^2 u}{d\theta^2}.$$

Next, we substitute into Equation (3) and get

$$-\frac{L^2 u^2}{P^3}\frac{d^2 u}{d\theta^2} - \left(\frac{P}{u}\right)\left(\frac{Lu^2}{P^2}\right)^2 = -\frac{L^2 u^2}{P^3}.$$

Dividing by the common factor of $-\dfrac{L^2 u^2}{P^3}$ results in the inhomogeneous second-order differential equation:

$$\frac{d^2 u}{d\theta^2} + u = 1.$$

The general solution of this has the form

$$u = u(\theta) = 1 + e\cos(\theta - \theta_0),$$

where e and θ_0 are constants which can be determined from the initial conditions. In terms of the polar radius r, this becomes

$$r = \frac{P}{1 + e\cos(\theta - \theta_0)}, \tag{5}$$

which gives a circle when $e = 0$, an ellipse when $|e| < 0$, a parabola when $|e| = 1$, and a hyperbola when $|e| > 1$ (see Exercise 1, below). Note that the coordinate system can be chosen so that $\theta_0 = 0$, (which puts the perihelion[1] on the positive x-axis) giving the equation of the form

$$r = \frac{P}{1 + e\cos\theta}, \tag{6}$$

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[1] The **perihelion** is the point of closest approach of the orbiting planet to the Sun.

from which Keplar's first law follows.

Determining the Eccentricity

In this section we take a closer look at Equation (5) and determine the eccentricity e in terms of the initial data. We imagine a planet of mass M and a nearby massive object—call it an asteroid—which has mass m. As we have seen, the trajectory of the asteroid is a conic section; we assume a coordinate system which places the planet at a focus. Assume that at time $t = 0$ we measure the distance of the asteroid to the plant to be r_0 and that

Figure 3: Initial configuration

the corresponding speed of the asteroid is v_0. Assume that relative to our coordinate system, the initial location of the asteroid corresponds to the angle $\theta(0) = \alpha$ and that the angle between the asteroid's velocity vector and the position vector is ϕ (as indicated) in Figure 3.
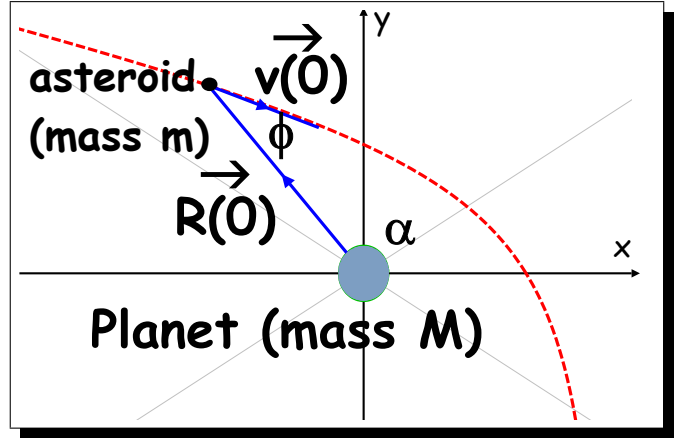
If we read Equation (5) at time $t = 0$, we get

$$r_0 = \frac{P}{1 + e\cos(\alpha - \theta_0)}. \tag{7}$$

It will be convenient to introduce the new constant $p = P/r_0$, and so Equation (7) implies that

$$1 = \frac{p}{1 + e\cos(\alpha - \theta_0)},$$

which gives

$$1 + e\cos(\alpha - \theta_0) = p. \tag{8}$$

Next, we differentiate Equation (5) with respect to t; noting the Chain Rule and setting $t = 0$, this leads to

$$v_0\cos\phi = \left.\frac{dr}{dt}\right|_{t=0} = \frac{Pe\sin(\alpha - \theta_0)}{[1 + e\cos(\alpha - \theta_0)]^2} \cdot \left.\frac{d\theta}{dt}\right|_{t=0} = \frac{ev_0\sin\phi\sin(\alpha - \theta_0)}{p}.$$

Therefore,

$$\cot\phi = \frac{e\sin(\alpha - \theta_0)}{p},$$

and so
$$e \sin(\alpha - \theta_0) = p \cot \phi. \tag{9}$$
Applying Equations (8) and (9) leads immediately to
$$e^2 = e^2 \cos^2(\alpha - \theta_0) + e^2 \sin^2(\alpha - \theta_0) = (p-1)^2 + p^2 \cot^2 \phi$$
which simplifies quickly to
$$e^2 = p^2 \csc^2 \phi - 2p + 1. \tag{10}$$

From the above, we get a useful trichotomy for the eccentricity e:

Ellipse ($e^2 < 1$): Equation (10) reduces in this case to
$$p^2 \csc^2 \phi - 2p + 1 < 1 \iff$$
$$p \csc^2 \phi < 2.$$

However, recall that
$$P = \frac{L^2}{GM}, \quad \text{where} \quad L = r_0^2 \frac{d\theta}{dt}\Big|_{t=0} = r_0 v_0 \sin \phi,$$
and so
$$p \csc^2 \phi = \frac{r_0 v_0^2}{GM}.$$
That is to say, an **elliptical orbit** will occur precisely when
$$r_0 v_0^2 < 2GM. \tag{11}$$

Parabola ($e^2 = 1$): Similarly with the above, this will occur when
$$r_0 v_0^2 = 2GM. \tag{12}$$

Hyperbola ($e^2 > 1$): This is when
$$r_0 v_0^2 > 2GM. \tag{13}$$

Exercise 1.

(a) Using the equations $x = r \cos \theta$, $y = r \sin \theta$, show that Equation (6) can be expressed in cartesian coordinates as
$$\left(x + \frac{Pe}{1 - e^2} \right)^2 + \frac{y^2}{1 - e^2} = \frac{P^2}{(1 - e^2)^2}, \quad \text{if} \quad e \neq 1,$$
and as
$$y^2 + 2Px = P^2 \quad \text{if} \quad e = 1.$$
From this, conclude that one obtains an ellipse if $e^2 < 1$, a parabola if $e^2 = 1$, and a hyperbola if $e^2 > 1$.

(b) Given that $e^2 < 1$, show that the semi-major axis of the ellipse has length $P/(1 - e^2)$ and the semi-minor axis has length $P/\sqrt{1 - e^2}$.

Exercise 2: Calculations of Eccentricities of Planetary Orbits about the Sun

The Earth's orbit

We itemize some estimates of the important constants:

- Distance at perihelion: $r_0 \approx 1.471 \times 10^8$ km.

- Linear velocity at **perihelion**[2]: $v_0 \approx 30.29$ km/sec.

- Universal gravitational constant: $G \approx 6.67 \times 10^{-11}$ N $\cdot$ m^2/kg^2

- Mass of Sun: $\approx 1.99 \times 10^{30}$ kg.

Using Equation (10), compute the eccentricity of the Earth's orbit about the Sun.[3]

Mars's orbit

The important constants are given below:

- Distance at perihelion: $r_0 \approx 2.066 \times 10^8$ km.

- Linear velocity at perihelion: $v_0 \approx 26.50$ km/sec.

Compute the eccentricity of Mars's orbit about the Sun.[4]

Mercury's orbit about Sun

The important constants are given below:

- Distance at perihelion: $r_0 \approx 4.600 \times 10^7$ km.

- Linear velocity at perihelion: $v_0 \approx 58.98$ km/sec.

Compute the eccentricity of Mars's orbit about the Sun.[5]

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[2] This is the point at which the Earth is closest to the Sun.
[3] You should get $e \approx 0.0167$, showing that the Earth's orbit is very close to a perfect circle.
[4] Here, $e \approx 0.0935$, showing that the Mars's orbit is more elliptical that that of the Earth.
[5] Here, $e \approx 0.2056$, making Mercury's orbit the most elliptical among the planets. (Actually Pluto's orbit is more elliptical, with $e \approx 0.248$, but in 2006 Pluto was downgraded from a "planet" to a "dwarf planet.")

Exercise 3: Minimum and Maximum Orbital Speeds

In this guided exercise you'll show that for the orbital model given by Equation (5), the body's maximum speed occurs precisely at its **perigee** (point of closest approach to the central mass), and when the orbit is elliptical, its minimum speed occurs at its **apogee** (point of maximum distance from the central mass).

Step 1. Using Equation (2), show that the speed is given by

$$v^2 = \left(\frac{dr}{dt}\right)^2 + r^2 \left(\frac{d\theta}{dt}\right)^2.$$

Step 2. Use Equation (5) to write

$$r = \frac{P}{1 + e\cos\theta}.$$

Also, you know that $r^2 \dfrac{d\theta}{dt}$ is a constant, which we earlier denoted by L:

$$\frac{d\theta}{dt} = \frac{L}{r^2}.$$

Combine this to get

$$v^2 = \left(\frac{L^2}{P^2}\right)\left[e^2 \sin^2\theta + (1 + e\cos\theta)^2\right].$$

Step 3. Show that

$$2v\frac{dv}{dt} = \text{constant} \times \frac{\sin\theta}{r^2}.$$

Step 4. Finish the proof that the maximum speed occurs at the perigee and the minimum speed occurs at the apogee (if any).[6]

Exercise 4: The Moon's Orbit about the Earth

You are given that the Moon's distance from Earth at perigee is about 364,397 km and its distance to the Earth at apogee is about 406,731 km. The mass of the Earth is $\approx 5.98 \times 10^{24}$ kg. Given this information, compute the eccentricity of the moon's orbit and its minimum and maximum linear speeds.[7]

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[6]Note that $L^2/P^2 = GM/P$; therefore the velocity at perigee is

$$v_{\max} = (1 + e)\sqrt{\frac{GM}{P}},$$

and the velocity at apogee (if any) is

$$v_{\min} = (1 - e)\sqrt{\frac{GM}{P}}.$$

[7]Simple algebra leads to

$$e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}} \approx 0.055.$$

Exercise 5. On the earth an astronomer spots a very slow-moving asteroid at time $t = 0$ moving at a speed of roughly $v_0 \approx 42$ m/sec and whose initial distance from the earth is approximately $r_0 \approx 7.5 \times 10^8$ km. Assume at the time of observation the angle ϕ between the velocity vector and $-\overrightarrow{R}$ (as shown) is approximately $15°$, and recall that the mass of the earth is $M \approx 5.98 \times 10^{24}$ kg. The value of the universal gravitational constant is roughly
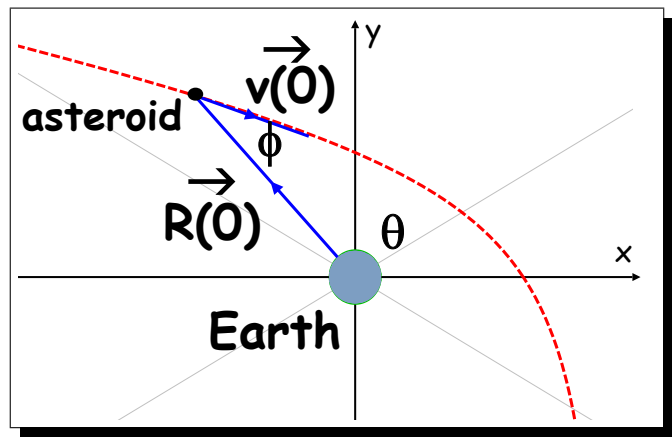

Figure 4: Asteroid initial conditions

$$G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2,$$

and the initial angle relative to the coordinate system indicated in Figure 4 is $\theta(0) \approx 120°$.

(i) Will this asteroid be captured into an elliptical orbit about the Earth?[8]

(ii) Compute the eccentricity of the asteroid's orbit relative to Earth.[9]

(iii) Relative to Figure 4 and the model

$$r = \frac{P}{1 + e\cos(\theta - \theta_0)},$$

determine the angular shift θ_0 and plot the trajectory of the asteroid.[10]

(iv) Compute the speed of the asteroid at its perigee. (Just use the result for v^2 in Step 2 of Exercise 2, above.)[11]

Exercise 6. Prove Kepler's Third Law, namely, that *the square of the orbital period (for an elliptical orbit) is proportional to the cube of the semi-major*

To compute the corresponding velocities, just use the equation

$$r = \frac{P}{1 + e\cos\theta};$$

the perigee occurs at $\theta = 0$, and $P = r_{\min}^2 v_{\max}^2/GM$. Put this together and get $v_{\max} = 1.074$ km/sec. Likewise, get $v_{\min} = 0.963$ km/sec.

[8]No, because
$$r_0 v_0^2 \approx 1.32 \times 10^{15} > 8 \times 10^{14} = 2GM.$$

[9]Using Equation (10), get $e \approx 1.14$, which dictates a hyperbolic orbit.
[10]From Equation (8), get $\theta(0) - \theta_0 \approx 133°$, and so $\theta_0 \approx -13°$.
[11]The constant L is given by $L = r_0 v_0 \sin\phi$, which implies that $L^2 = PGM$. Therefore, the velocity $v_{\max}$ at perigee is determined through
$$v_{\max}^2 = \left(\frac{GM}{P}\right)(1 + e)^2 \implies v_{\max} \approx 105 \text{ m/sec}$$

axis.

To do this, just carry out the indicated steps.

Step 1. Denoting by T the orbital period, note that the area of the ellipse swept out by the orbiting body is

$$A = \frac{1}{2} \int_{t=0}^{t=T} r^2 \frac{d\theta}{dt} \, dt.$$

However, we've seen that $r^2 \dfrac{d\theta}{dt}$ is a constant, denoted L. Therefore, the above area is given by $A = \frac{1}{2} LT$.

Step 2. Denoting by a the semi-major axis and by b the semi-minor axis, the area of the corresponding ellipse can be expressed by the integral

$$A = \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} \, dx = ab\pi.$$

Step 3. Recall from Analytical Geometry that an ellipse with semi-major axis a and eccentricity e has semi-minor axis $b = a\sqrt{1 - e^2}$. (See also Exercise 1, above.) Conclude from Steps 1 and 2 that

$$T = \frac{2A}{L} = \frac{2a^2 \sqrt{1 - e^2}\pi}{L}.$$

Step 4. Using the equation

$$r = \frac{P}{1 + e \cos\theta}, \quad P = \frac{L^2}{GM},$$

together with

$$2a = r_{\min} + r_{\max} = \frac{P}{1+e} + \frac{P}{1-e}$$

leads immediately to

$$L^2 = a(1 - e^2)GM.$$

Step 5. Conclude by showing that $\dfrac{T^2}{a^3} = \dfrac{4\pi^2}{GM}$.

Exercise 7: A Paradox?

As we Suppose that we return to the Earth-Moon system of Exercise 4, where the Moon's orbit about the Earth was determined to be elliptical, having eccentricity $e \approx 0.055$. Furthermore, a moment's thought reveals that relative to observers

on the Moon, **the Earth can be said to orbit about the Moon** in an elliptical orbit with the same eccentricity and the same velocities at perigee and apogee. However, suppose that we adopt a coordinate system which places the Moon at the origin and the positive x-axis in the direction of the Earth's perigee relative to the Moon. Then the equation of motion predicting the Earth's motion relative to the Moon should be

$$r = \frac{P}{1 + e \cos \theta},$$

where $P = \dfrac{r_{\min}^2 v_{\max}^2}{Gm}$, and where m is the mass of the Moon and is

$$m \approx 7.36 \times 10^{22} \text{ kg.}$$

However, using $r_{\min} \approx 3.64 \times 10^5$ km and $v_{\max} \approx 1.05$ km/sec, the above equation returns an eccentricity of

$$e = \frac{P}{r_{\min}} - 1 \approx 80.8,$$

which isn't the eccentricity of an ellipse at all! What gives?