KNIGHTS AND SPIES: A SPECIAL CASE OF CONJECTURE 5.1

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My paper [1] includes the following conjecture.

Conjecture 5.1. Let $g : \mathbf{N} \to \mathbf{N}$ be such that $g(\ell) > 2\ell$ for all $\ell \in \mathbf{N}$. Let $0 < \sigma \leq 1$. There is a questioning strategy which, provided that ℓ is sufficiently large, guarantees to use at most $g(\ell) + \ell - 1$ questions to find all the identities in an $g(\ell)$-person room known to contain at most ℓ spies, and in fact containing exactly $\lfloor \sigma\ell \rfloor$ spies, and will on average use at most $g(\ell) + 3\ell/4$ questions.

When $g(\ell) \leq \ell^2$, this conjecture can be proved by modifying the Spider Interrogation Strategy presented in §2 of [1]. We give the required changes in outline.

Step 1. Ask person 1 about person 2, then person 2 about person 3, and continue in this manner, until either we meet an accusation, or we have asked ℓ questions. In the latter case, person $\ell + 1$ must be a knight. If we ask him about everyone else in the room, then we find everyone's identity in $n + \ell - 1$ questions. Moreover, if we begin by asking him about person 1 then, if person 1 transpires to be a knight, the resulting cycle in the question graph implies that the first $\ell + 1$ people are all knights. A further $n - (\ell + 1)$ questions find all the remaining identities, giving a total of just n questions.

Suppose instead that we meet an accusation when person t accuses person $t + 1$. If $t = 1$, then we have not yet departed from the normal Spider Interrogation Strategy. If $t > 1$, then treat person t as a candidate who has been supported by $t - 1$ people, and accused by one, and continue to question fresh people about him as in Step 1 of the unmodified strategy. Should he be rejected, choose a new candidate and continue to follow Step 1 of the unmodified strategy; if the resulting 'spider' in the question graph contains $2b$ people, then at least b of them are spies, and so the threshold for acceptance of the new candidate is $\ell - b$.

Steps 2, 3 and 4. These are analogous to the unmodified strategy. The proof of Proposition 2.1 in [1] can readily be adapted to show that whether person t is accepted (after accusing person $t+1$), or rejected, $n+\ell-1$ questions suffice to find everyone's identity. Figure 1 below shows an illustrative example.

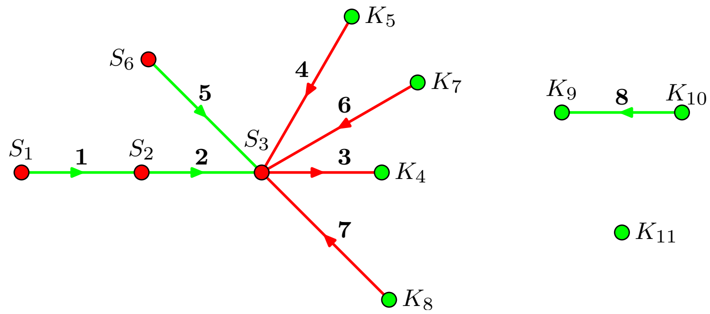

Figure 1. The end of Step 1 of the modified Spider Interrogation Strategy in an 11 person room with $\ell = 5$, in which spies lie in all their answers. The first candidate S_3 is rejected, and the second candidate K_9 is accepted. In Step 2, the knight K_9 will be asked about S_3 and K_{11}, and in the modified version of Step 3, he will be asked about his fellow knights, K_4, K_5, K_7, K_8 and K_{10}. The full 15 questions are required.

The event that none of the first $\ell+1$ people in the room is a spy has probability at least

$$p_\ell(n) = \Bigl(1 - \frac{\ell}{n-\ell}\Bigr)^{\ell+1}.$$

For fixed ℓ, the lower bound $p_\ell(n)$ is an increasing function of n. Moreover,

$$q(\ell) = p_\ell(\ell^2) = \Bigl(1 - \frac{1}{\ell-1}\Bigr)^{\ell+1}$$

is an increasing function of ℓ for $\ell \geq 2$, tending to $1/\mathrm{e}$ as $\ell \to \infty$. Calculation shows that $q(9) \geq 1/4$, and hence $p_\ell(n) \geq 1/4$ whenever $9 \leq \ell \leq \sqrt{n}$. This proves Conjecture 1 in the special case when $g(\ell) \leq \ell^2$.

References

[1] M. Wildon, Knights, spies, games and ballot sequences. *To appear in Disc. Math.*