KNIGHTS, SPIES, GAMES AND BALLOT SEQUENCES

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ABSTRACT. This paper solves the Knights and Spies Problem: In a room there are n people, each labelled with a unique number between 1 and n. A person may either be a knight or a spy. Knights always tell the truth, while spies may lie or tell the truth as they see fit. Each person in the room knows the identity of everyone else. Apart from this, all that is known is that strictly more knights than spies are present. Asking only questions of the form: 'Person i, what is the identity of person j?', what is the least number of questions that will guarantee to find the true identities of all n people? We present a questioning strategy that uses slightly less than $3n/2$ questions, and prove that it is optimal by solving a related two-player game. The performance of this strategy is analysed using methods from the famous ballot-counting problem. We end by discussing two questions suggested by generalisations of the original problem.

1. INTRODUCTION

In this paper we solve the *Knights and Spies Problem*: In a room there are n people, each labelled with a unique number between 1 and n. A person may either be a *knight* or a *spy*. Knights always tell the truth, while spies may lie or tell the truth as they see fit. Each person in the room knows the identity of everyone else. Apart from this, all that is known is that strictly more knights than spies are present. Asking only questions of the form:

'Person i, what is the identity of person j?',

what is the least number of questions that will guarantee to find the true identities of all n people?

Despite its apparently recreational character, the Knights and Spies Problem proves to be surprisingly deep. It is rare to find such an easily stated problem that can challenge and be enjoyed by professionals and amateurs alike. The following remarks introduce some basic ideas and should clarify its statement.

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1.1. **Preliminary remarks.** A simple, if inefficient, questioning strategy suffices to find everyone's identity. Assume for the moment that n is even, with say $n = 2m$. Given a person i, if we ask the remaining $2m-1$ people to state person i's identity, then the majority opinion will be correct. For otherwise m or more people have lied, and since only spies can lie, all these people must be spies. With a small extension to deal with ties in the case when n is odd, this gives us a strategy that finds everyone's identity in $n(n-1)$ questions.

We may refine this strategy by noting that anyone who ever holds a minority view is a liar. Such people can be immediately identified as spies, and then ignored in future questions. Moreover, once we have found a knight, we may bombard him with questions to find all the remaining identities. However, even with these improvements, the number of questions required in the worst case is still quadratic in n.

When this strategy is followed, the spies are at their most obstructive when they always tell the truth. This phenomenon will be seen in other contexts below. We may assume, however, that a spy will lie if asked to state his own identity, and so while it is permitted by the rules, there can be no benefit in asking a person about himself. Similarly, there can be no benefit in asking the same question to any person more than once.

Before reading any further, the reader is invited to find a questioning strategy that will use at most Cn questions for some constant C. A hint leading to a strategy for which $C = 2$ is given in this footnote.[1] The optimal C and the answer to the problem are revealed in the outline below.

1.2. **Outline.** We shall solve the more general problem, where it is given that at most ℓ spies are present for some ℓ with $1 \leq \ell < n/2$. We begin in §2 by describing the *Spider Interrogation Strategy*, which guarantees to find everyone's identity using at most

$$n + \ell - 1$$

questions. If, as in the original problem, all we know is that knights are strictly in the majority, then $\ell = \lfloor (n-1)/2 \rfloor$, and so the maximum number of questions asked is $f(n)$, where f is the function defined by

$$\begin{aligned} f(2m-1) &= 3m - 3 \\ f(2m) &= 3m - 2. \end{aligned}$$

In §3 we prove that any questioning strategy will, in the worst case, require at least $n+\ell-1$ questions. Hence the answer to the original problem is that, provided

[1] There is an inductive strategy that starts by putting people into pairs, leaving one person out if n is odd, and then asking each member of each pair about the other.

$n \geq 3$, the smallest number of questions that can guarantee success is $f(n)$. Since

$$0 \leq 3n/2 - f(n) \leq 2$$

for all natural numbers n, the lowest possible value for the constant C is $3/2$. The proof in §3 is presented as an optimal strategy for the second player in the two-player game in which the first player poses questions (in the standard form), and the second supplies the answers ('knight' or 'spy'), with the aim of forcing his opponent to ask at least $n + \ell - 1$ questions before he can be certain of everyone's identity. This game provides a setting for all the problems considered in this paper.

In §4 we consider the behaviour of the Spider Interrogator Strategy against spies that always lie when asked about a knight. (This situation arises naturally in the context of the strategies considered in §3.) We show that usually fewer than $n+\ell-1$ questions are required, and determine the probability distribution of the number of questions asked; remarkably it is independent of how the spies behave when they are asked about one another. The proof is combinatorial and uses two lemmas related to the famous ballot-counting problem (see [3, §III.1]).

It is natural to ask whether there is a questioning strategy which never uses more than $n + \ell - 1$ questions, and will with reasonable probability use fewer, no matter how cleverly the spies answer. (Of course, given the result of §3, there is inevitably a non-zero probability that the full number of questions will be required.) We end by briefly discussing this problem. We also state a further open problem suggested by varying the rules of the game introduced in §3 of this paper.

2. The Spider Interrogation Strategy

The *Spider Interrogation Strategy* has four steps: its first, in which we hunt for someone who we can guarantee is a knight, is the key to its workings. We suppose that at most ℓ of the n people in the room are spies, where $1 \leq \ell < n/2$.

Step 1. Choose any person as a *candidate*. Repeatedly ask new people about the candidate until *either*

(a) strictly more people have said that the candidate is a spy than have said that he is a knight, *or*

(b) ℓ people have said that the candidate is a knight.

If we end in case (a), with the candidate accused by a different people, then he must have been supported by $a - 1$ different people. Whatever his true identity, it is easily checked that at least a of the $2a$ people involved are spies. Hence if we reject the candidate, ignore all $2a$ of the people involved so far, and replace ℓ with $\ell - a$, then we may repeat Step 1 with a smaller problem. Eventually, since spies are in a strict minority, we must finish in case (b). The successful candidate is supported by ℓ people, so must be a knight.

Step 2. Let person k be the knight found at the end of Step 1. All future questions will be addressed to him. In this step, use him to identify each person who has not yet been involved in proceedings, and also each of the rejected candidates from case (a) of Step 1.

Step 3. Let persons $m_1, \ldots, m_t$ be the rejected candidates whose identities were determined in Step 2. Suppose that person m_i was accused by a_i people.

(a) If person m_i is a knight, then the a_i people who accused him are spies. Identify the $a_i - 1$ people who supported him.

(b) If person m_i is a spy, then the $a_i - 1$ people who supported him are spies. Identify the a_i people who accused him.

Step 4. Finally, identify each person who supported the knight k's candidacy. Since the people who accused the knight k must be spies, everyone's identity is now known.

It is useful to represent the progress of the Spider Interrogation Strategy by a coloured digraph on the set $\{1, 2, \ldots, n\}$ in which we draw an edge from vertex i to vertex j if person i has been asked about person j, and colour it according to person i's answer. We shall refer to such a graph as a *question graph*. Figure 1 overleaf shows a question graph after Step 1 of the Spider Interrogation Strategy. Its characteristic structure gives the Spider Interrogation Strategy its name.

It is not hard to show that a questioner following the Spider Interrogation Strategy asks at most $n + \ell - 1$ questions. In fact we can easily prove something more precise.

Proposition 2.1. *The total number of questions asked by a questioner following the Spider Interrogation Strategy is*

$$n + \ell - 1 - r$$

where r is the number of knights rejected as candidates in its first step.

Proof. After Step 2 is complete, the underlying graph of the question graph is a tree. Therefore $n - 1$ questions have been asked by this point. The number of questions asked in Step 3 is $a_1 + \cdots + a_t - r$. The knight k was accepted after $\ell - (a_1 + \cdots + a_t)$ people supported him, hence the total number of questions asked in Steps 3 and 4 is $\ell - r$. □

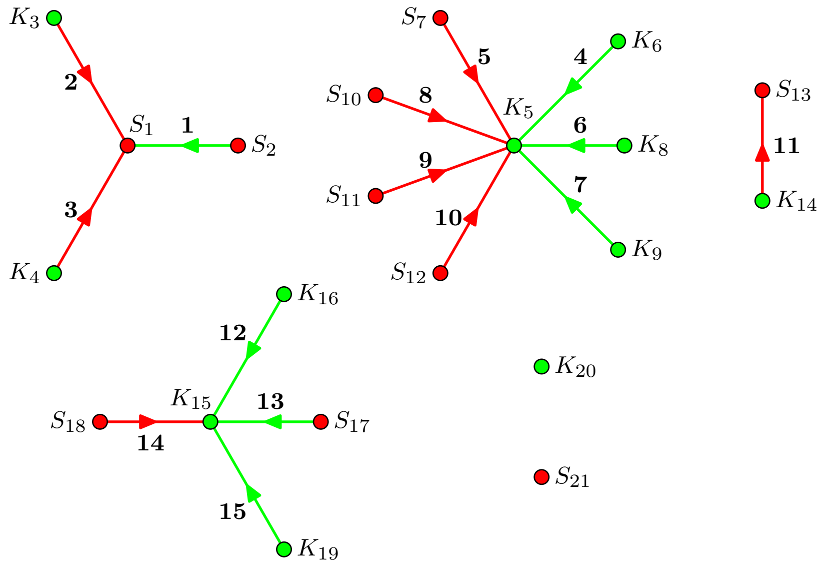

Figure 1. The question graph at the end of Step 1 of the Spider Interrogation Strategy in a 21-person room with $\ell = 10$. The labels S_i and K_i are used to indicate the nature of the person represented by vertex i. Green arrows show supportive statements and red arrows show accusations. Questions are numbered in bold. The candidates are S_1 (rejected), K_5 (rejected), S_{13} (rejected) and K_{15} (successful). Spies are assumed to lie in all their answers. except for S_{17}, who we have supposed answers truthfully when asked about K_{15}. All future questions will be addressed to the knight K_{15}. For instance, in Step 2 he will be asked about S_1, K_5, S_{13}, K_{20} and S_{21}. The total number of questions asked is 29.

An interesting feature of the Spider Interrogation Strategy, already visible in Figure 1, is that it guarantees that each spy in the room will be asked at most one question.

3. A lower bound

In this section, we shall show that any questioning strategy, will, in the worst case, require at least $n + \ell - 1$ questions to find everyone's true identity.

The difficulty we face in proving this result is that we must somehow take into account *every* possible questioning strategy that can be employed, irrespective of how bizarre it might seem. This is much the same problem that confronts a player of a game such as chess or *go*, and so it is perhaps not surprising that it is helpful to think of our problem in this context.

3.1. **A mathematical game.** The game of *'Knights and Spies'* is played between two players: an *Interrogator* and a *Spy Master*. At the start of the game the players agree on values for the usual parameters n and ℓ, with as ever $1 \leq \ell < n/2$.

In a typical turn the Interrogator poses a question (in the standard form) to the Spy Master. The Spy Master must supply an answer ('knight' or 'spy') which is consistent with there being at most ℓ spies present in the room. It is worth emphasising that, while the Spy Master's answers will indirectly determine the arrangement of knights and spies in the room, the Spy Master is not committed (even privately) to any particular arrangement when the game begins.

The Interrogator's aim is, of course, to determine everyone's identity. The Spy Master acts as the agent of malign fate; his aim is to force the Interrogator to experience the worst-case scenario for his chosen questioning strategy.

If, at the beginning of a turn, the Interrogator believes that he is certain of everyone's identity, he may *claim* by giving the full set of people who he believes are spies. The Spy Master must then either *refute* his claim, by exhibiting a different set that is also consistent with the questions and answers so far, or agree that the secret is out. The Interrogator wins if he makes a successful claim before turn $n + \ell$, and draws if he makes a successful claim at the start of turn $n + \ell$ (after having asked $n + \ell - 1$ questions). In any other event victory goes to the Spy Master. The computer program *Gamechecker* described in the appendix to this paper may be used by players of the game to enforce these rules.

Our required result is equivalent to the following theorem.

Theorem 3.1. *The Spy Master has a strategy that ensures the Interrogator cannot claim before he has asked* $n + \ell - 1$ *questions.*

We refer the reader to [5, §10.1] for a formal axiomatisation of two-player games that is capable of expressing Theorem 3.1.

In the following two sections we give two different proofs of Theorem 3.1 that each illustrate different features of the game we have introduced. When describing strategies for the Spy Master, we shall assume for simplicity that the Interrogator never asks anyone to state his own identity, or repeats a question verbatim; the discussion in §1.1 tells the Spy Master how to reply to such questions, and shows that this is not a significant restriction.

3.2. **The Knight Hiding Strategy.** This strategy has two phases.

Phase 1. Answer the first $\ell - 1$ questions posed by the Interrogator with blanket accusations. Let G be the question graph after the first $\ell - 1$ questions. Suppose that the underlying graph of G is the union of the connected components $G_1, \ldots, G_c$.

Phase 2. The Spy Master should now hide one knight in each connected component of G. He does this by answering any question addressed to a person j in the component G_i with the answer 'spy', *unless* in Phase 2 of the game the Interrogator has already asked about everyone else in G_i; in this case person j becomes the *hidden knight* in the component G_i, and the Spy Master truthfully answers 'knight'.

An example game in which $n = 12$ and $\ell = 5$ is shown in Figure 2 below. The graph G has connected components $\{1, 2, 3\}, \{4, 5\}, \{6\}, \{7\}, \dots, \{12\}$. The Interrogator can be sure after 16 questions that the only spies present are persons 2, 3 and 5, but is unable to claim any earlier; the game therefore ends in a draw. Note that in Phase 2, the Spy Master always supports people in singleton components of G; the hidden knights in the larger components are persons 1 and 4.

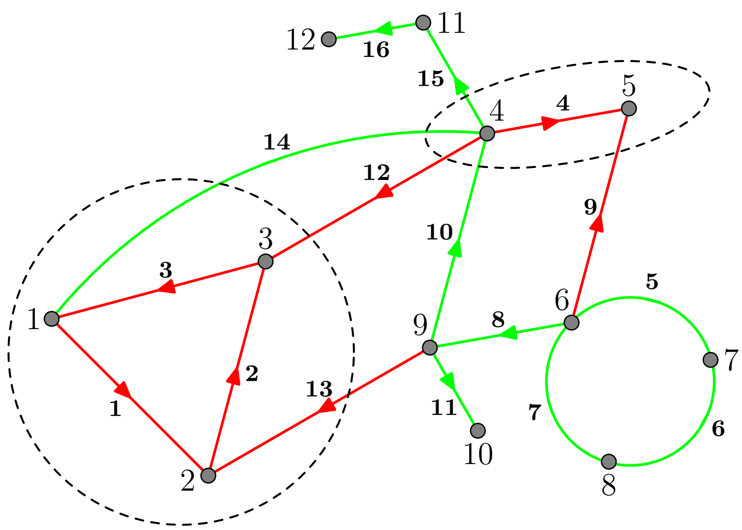

Figure 2. A game in a 12 person room with $\ell = 5$. The Spy Master adopts the Knight Hiding Strategy, and holds the Interrogator to a draw. Questions are numbered in bold. The non-singleton connected components of the subgraph G are enclosed with dashes.

The following proposition shows that the Interrogator's time consuming search through the singleton components of G for hidden spies, and through the larger components of G for hidden knights, is unavoidable.

Proposition 3.2. *If the Spy Master follows the Knight Hiding Strategy then, at every point in the game, there is a subset of people that can consistently be the set of spies in the room. Moreover, at the beginning of each turn t with $t \leq n + \ell - 1$, there are two different such subsets.*

Proof. Suppose that the game is at the beginning of turn t. Since extra questions can only increase the requirements a consistent assignment of identities to the people in the room must satisfy, we may assume without loss of generality that $t \geq \ell - 1$. Hence the subgraph G is defined and Phase 1 of the game has ended.

For each component G_i of G, if in Phase 2 of the game the Interrogator has already asked about everyone in G_i, then let person k_i be the hidden knight in G_i. Otherwise choose for k_i any person in G_i who has not yet been asked about. Let

$$S = G \setminus \{k_1, \ldots, k_c\},$$

and let K be the complement,

$$K = \{k_1, \ldots, k_c\}.$$

Let $k \in K$ and let $y \in \{1, 2, \ldots, n\}$. If the Spy Master has told the Interrogator that person k supports person y, then this question must have occurred in Phase 2 of the game, and $y \in \{k_1, \ldots, k_c\}$. Similarly, if the Spy Master has told the Interrogator that person k accuses person y, then $y \in S$. Hence, provided that S is not too large, the Spy Master's answers are consistent with S being the full set of spies. If the component G_i contains v_i people and has e_i edges then the number of questions asked in Phase 1 of the game is $e_1 + \cdots + e_c = \ell - 1$. By a standard result, $v_i - 1 \leq e_i$ for each i, and hence

$$\begin{aligned} |S| &= (v_1 - 1) + (v_2 - 1) + \cdots + (v_c - 1) \\ &\leq e_1 + \cdots + e_c \\ &= \ell - 1. \end{aligned}$$

Thus the size of our claimed set of spies is safely below the upper bound of ℓ.

Now suppose that $t \leq n + \ell - 1$. At most $n - 1$ questions have been asked in Phase 2 of the game, so there is some person, say person x, who has not been asked about in this phase. If x belongs to the connected component G_i of G then we may assume that we chose $k_i = x$. Let $S^\star = S \cup \{x\}$. By our choice of x, person x has never been supported by anyone in the room. Hence it is consistent that he is a spy. Since $|S^\star| = |S| + 1 \leq \ell$, it is consistent that $S^\star$ is the set of spies. $\square$

It follows from the first part of Proposition 3.2 that the Spy Master can adopt the Knight Hiding Strategy without breaking the rules of the game. The second part implies that the Interrogator will be unable to claim before he has asked $n + \ell - 1$ questions. Theorem 3.1 is an immediate corollary.

3.3. **The Knavish Dividing Strategy.** In this section we outline an alternative drawing strategy for the Spy Master which has the interesting property that the spies lie in all their answers. We shall say that such spies are *knavish*. (The Spy

Master may, of course, depart from this self-imposed restriction in order to refute a premature claim from the Interrogator.) Again the strategy has two phases.

Phase 1: Answer all the Interrogator's questions with accusations, unless this will create an (undirected) cycle of odd length in the subgraph of the question graph formed by the accusations so far; in this case, support. This rule ensures that the graph of accusations is always bipartite. Phase 1 ends at the first question when, were the Spy Master to accuse, the size of the smallest possible part in a bipartition of the graph of accusations would reach ℓ or more.

Phase 2: The Spy Master supports on the question that ended Phase 1. He then commits himself to putting knavish spies in the smaller part, and knights in the larger part, of each connected bipartite component of the graph of accusations. In bipartite components in which the parts have equal sizes, he waits for a new question involving a person in this component, and then accuses.

The situation part-way through an example game is shown in Figure 3 below.

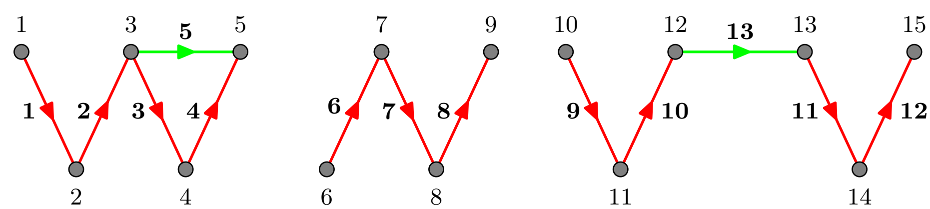

Figure 3. The Knavish Dividing Strategy in a 15 person room with $\ell = 7$. Phase 2 has begun with question 13. The Spy Master will play to make persons $1, 3, 5, 10, 12, 13$ and 15 knights. If, for example, question 14 involves person 6, then the Spy Master will accuse, and then play to make persons 7 and 9 knights.

We shall denote a connected bipartite component in the (undirected) graph of accusations with parts of sizes α and β with $\alpha \geq \beta$ by $\alpha \,|\, \beta$; a component $\alpha \,|\, \beta$ is said to be *balanced* if $\alpha = \beta$. (The component containing a person not yet involved in proceedings is therefore $1 \,|\, 0$.)

It is easy to check that a component $\alpha \,|\, \beta$ in Phase 1 may have either β or $\beta + 1$ spies, unless the Interrogator has spent ℓ questions to build a cycle of length ℓ through the people making up the part of size α. We leave it to the reader to use this remark to show that the Interrogator will be unable to claim in Phase 1 before he has asked $n + \ell - 1$ questions.

The interesting case occurs when the Spy Master is forced to enter Phase 2. Suppose that this happens when the accusatory part of the question graph has components

$$\alpha_1 \,|\, \beta_1,\ \ldots,\ \alpha_d \,|\, \beta_d,\ \alpha \,|\, \beta,\ \alpha' \,|\, \beta'$$

and that the Interrogator asks a question which, were the Spy Master to accuse, would connect the component $\alpha \,|\, \beta$ to the component $\alpha' \,|\, \beta'$. Since the Spy Master is unable to accuse, neither of these components is balanced, and $\beta_1 + \cdots + \beta_d + \beta + \alpha' \geq \ell$. Hence

$$(\star) \qquad \alpha_1 + \cdots + \alpha_d + \alpha + \alpha' - e \geq \ell + 1$$

where e is the number of unbalanced components amongst $\alpha_1 \,|\, \beta_1, \ldots, \alpha_d \,|\, \beta_d$.

If the Spy Master made t supportive statements in Phase 1 then it takes the Interrogator at least

$$\alpha_1 + \cdots + \alpha_d + \alpha + \alpha' + (d - e) - t$$

questions to obtain a supportive answer from the Spy Master about each of the knights in the Interrogator's final successful claim. Unless all these answers have been obtained, the identity of one of these people will be ambiguous. It now follows from $(\star)$ that the total number of questions in Phase 2 is at least $\ell + 1 + d - t$. At the end of Phase 1, the graph of accusations has $d+2$ components. Hence the total number of accusations in Phase 1 is at least $n - (d+2)$, and the total number of questions is at least $n - (d+2) + t$. Therefore, over the entire game, the Interrogator asks at least

$$(n - d - 2 + t) + (\ell + 1 + d - t) = n + \ell - 1$$

questions. This proves that the Knavish Dividing Strategy holds the Interrogator to a draw, while ensuring that spies behave knavishly at all times.

3.4. **Final remarks on the game.** We end this section with two remarks on the game we have introduced, both with a hint of the paradoxical. Firstly, the author's experience is that most players expect to find it easier to play as the Spy Master than the Interrogator, but, to their surprise, find that after the first few games, the reverse is true. Since it is far from obvious that $n + \ell - 1$ questions suffice, this seems somewhat remarkable.

Secondly we note that both Knight Hiding and Knavish Dividing are optimal strategies (in the game-theoretic sense) since each guarantees to hold the Interrogator to $n + \ell - 1$ questions; given the existence of the Spider Interrogation Strategy, this is the best the Spy Master can hope for. This is not to say that they cannot be improved. Their common defect is that neither punishes bad play on the part of the Interrogator as harshly as is possible.

For example, in the game shown in Figure 2, the Interrogator's third question was in fact a game-losing blunder, for after it the Spy Master can safely extend Phase 1 of the game by one more question, forcing the Interrogator to ask 17 questions before he can claim. More generally, by modifying the Knight Hiding Strategy, the Spy Master can win any game in which the Interrogator's first $\ell - 1$ questions form an undirected cycle. It would be interesting to know if there are any other early plays by the Interrogator that can be punished.

4. Ballot sequences

We now return to the Spider Interrogation Strategy. Proposition 2.1 implies that a questioner following this strategy saves one question from the maximum of $n+\ell-1$ every time a knight is rejected as a candidate. No matter how the spies are arranged in the room, they can easily make sure this never happens, most simply by answering every question truthfully. If however the spies are constrained to lie when asked about a knight (as in the Knavish Dividing Strategy) then it is probable that fewer questions will be required.

Theorem 4.1. *Suppose that there are k knights and $s < \ell < n/2$ spies randomly arranged in an n-person room, and that the spies always answer 'spy' when asked about a knight. The probability that a questioner following the Spider Interrogation Strategy asks exactly q questions does not depend on how the spies answer when they are asked about one another. The expected number of questions saved is*

$$\frac{1}{\binom{k+s}{s}} \sum_{i=0}^{s-1} \binom{k+s}{i}.$$

We assume that $s < \ell$ in order to exclude the possibility that the questioner unmasks ℓ spies before the end of Step 4 of the strategy, and so can finish early.

It follows from Theorem 4.1 and Stirling's formula that in a large room in which knights are only just in the majority, a questioner following the Spider Interrogation Strategy can expect to ask about

$$\frac{3n}{2} - \sqrt{\frac{\pi}{8}}\sqrt{n}$$

questions. Another asymptotic result worth noting is that when $k = 2s$, the sum of binomial coefficients agrees in the limit with $\binom{3s}{s}$, and so the expected number of questions saved tends to one as s tends to infinity. This result may be proved by using the lower bound $\binom{3s}{s-j}/\binom{3s}{s-j+1} = (s-j+1)/(2s+j) \geq (1-\vartheta)/(2+\vartheta)$, valid for $j < \vartheta s$, to approximate the corresponding part of the binomial sum with a geometric series with ratio $1/2$.

Our proof of Theorem 4.1 depends on two combinatorial bijections (defined in §4.1 below), and does not give an explicit formula for the probabilities involved. Indeed, it seems unlikely that any simple such formula exists.

4.1. **Paths.** We shall represent the sequence of questions asked in Step 1 of the Spider Interrogation Strategy by a path in which we step up every time a knight is supported or a spy is accused, and down every time a knight is accused or a spy is supported. An initial step, which could be regarded as the candidate implicitly supporting himself, is taken whenever a new candidate is chosen.

Rather than end the path when a candidate is accepted, we instead question the remaining people about the accepted candidate. Thus everyone in the room will either have been a candidate or have been asked a question, and so our paths will always have exactly n steps. This extension of paths mimics Fermat's solution of the famous *Problème des Points* (see [1, page 300] for an accessible account). Figure 4 below shows an example path based on the 21-person room shown in Figure 1.

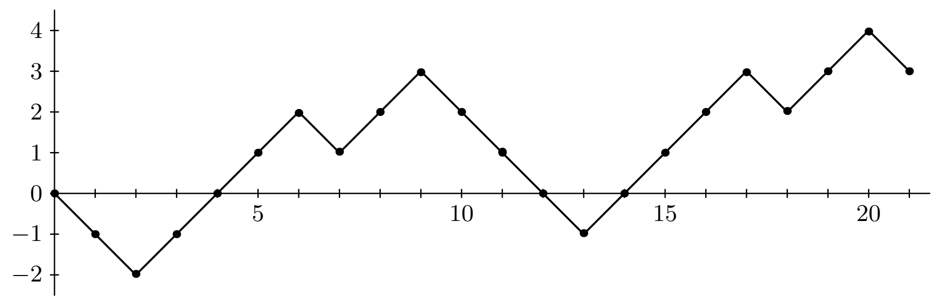

Figure 4. The path corresponding to Step 1 of the Spider Interrogation Strategy in the 21-person room shown in Figure 1, as modified by replacing the spy S_{17} (who in Figure 1 told the truth about K_{15}) with a knight. The final two steps correspond to extra questions asked to the knight K_{20} and the spy S_{21} about the successful candidate K_{15}. Spies lie in all their answers.

We say that a path *visits* m *from above at time* r if its height after r steps is m, and its rth step is downwards. Thus the path shown in Figure 4 visits 1 from above exactly twice, at times 7 and 11.

Lemma 4.2. *Let P be a path representing the questions asked in Step 1 of the Spider Interrogation Strategy. There is a bijection between visits of P to* 0 *from above and the knights rejected in this step.*

Proof. It suffices to prove that, once a candidate has been accepted, the path never returns to 0. This is left to the reader as a straightforward exercise. □

The next two lemmas, which can be stated without reference to our intended application to the Spider Interrogation Strategy, record two key probabilistic results on paths. We give each path with a fixed number of upsteps and downsteps the same probability.

Lemma 4.3. *Let $k \geq s$ and let $p \geq 0$. The probabilities that a path with k upsteps and s downsteps visits m from above exactly p times are equal for m in the range $-1 \leq m \leq k - s$.*

Proof. Let $0 \leq m \leq k - s$. We shall show that the probabilities agree for $m - 1$ and m. Let P be a path with k upsteps and s downsteps. Suppose that the first time P visits m is after step b, and that the last time P visits m is after step c. (Since $m \geq k - s$, the times b and c are well-defined.) Reflecting the part of P between b and c in the line $y = m$ gives a new path, P'. Figure 5 below shows this reflection when $m = 1$ for the path shown in Figure 4.

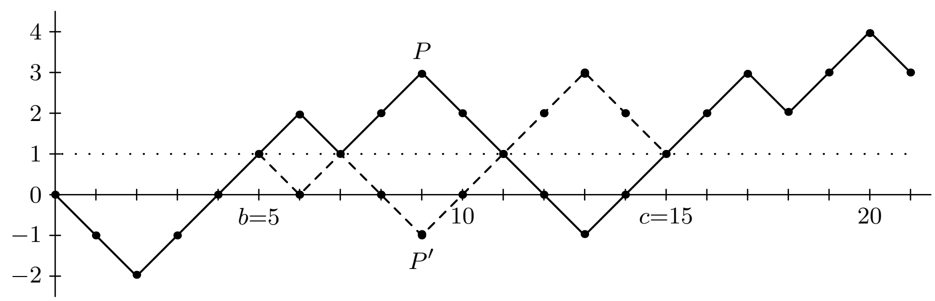

Figure 5. The path P first visits 1 after step 5, and last visits 1 after step 15. The path P' is the reflection of P in the line $y = 1$ between $b = 5$ and $c = 15$.

One easily sees that P visits m from above exactly as many times as P' visits $m - 1$ from above. Similarly P visits $m - 1$ from above exactly as many times as P' visits m from above. The result follows. □

Lemma 4.4. *Let $k \geq s$. The expected number of visits to -1 from above for a path with k upsteps and s downsteps is*

$$\frac{1}{\binom{k+s}{s}} \sum_{i=0}^{s-1} \binom{k+s}{i}.$$

Proof. Let $c(k, s)$ be the total number of times all paths with k upsteps and s downsteps visit -1 from above. We must prove that

$$c(k, s) = \sum_{i=0}^{s-1} \binom{k+s}{i}.$$

We work by induction on s. If $s = 0$ then it is impossible for any path to visit -1, so the result obviously holds in this case.

For the inductive step we use reflection in a different way than in Lemma 4.3; this is in fact the standard way it is used. (Feller [3, Chapter III], gives a good introduction to this reflection argument and its possible applications.) Let P be a path with k upsteps and s downsteps which visits -1 from above at least once. If P visits -1 for the first time after step d, then reflect the part of P between 0 and d in the line $y = -1$. This gives a new path $P^\star$ from $(0, -2)$ to $(k + s, k - s)$, as shown in Figure 6 below.

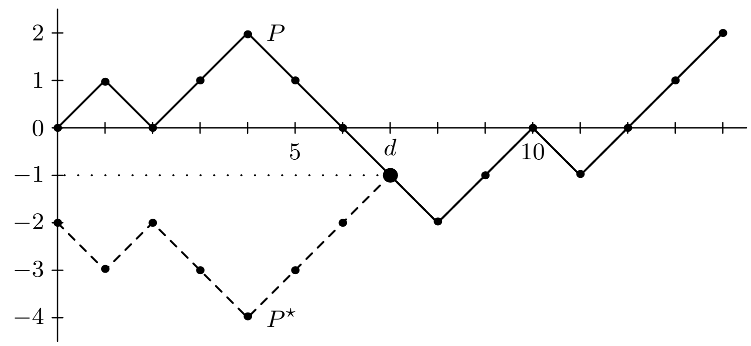

Figure 6. The path P, which visits -1 for the first time after step $d = 7$, is reflected to the path $P^\star$ starting at $(0, -2)$.

If P visits -1 from above exactly m times then $P^\star$ visits -1 from above exactly $m - 1$ times. Since there are $\binom{k+s}{s-1}$ possible paths $P^\star$ from $(0, -2)$ to $(k + s, k - s)$, each with $k + 1$ upsteps and $s - 1$ downsteps, we have

$$c(k, s) = \binom{k+s}{s-1} + t,$$

where t is the total number of times all paths from $(0, -2)$ to $(k + s, k - s)$ visit -1 from above. Each such path has $k + 1$ upsteps and $s - 1$ downsteps. Shifting upwards to $(0, 0)$ and applying Lemma 4.3, we see that $t = c(k + 1, s - 1)$. The lemma now follows by induction. □

The formula in the statement of the previous lemma appears in Engelberg [2, Equation (6)], where it is shown to enumerate the expected number of times a path with k upsteps and s downsteps (where $k > s$) crosses 0. It would be interesting to know if there is a bijective proof of this equidistribution. See also [4] for some related results on ballot sequences.

4.2. **Proof of Theorem 4.1.** We now apply our results on paths to prove Theorem 4.1. By Lemma 4.2, the first part of this theorem is implied by the $j = n$ case of the following claim. *Claim:* after $n - j$ steps, the probability that a path representing the sequence of questions asked in the course of the Spider Interrogation

Strategy visits 0 from above exactly p times is independent of how the spies answer when asked about one another in the remaining questions.

We shall prove the claim by induction on j, the $j = 0$ case being obvious.

Let $j \geq 1$. Let k' be the number of knights and let s' be the number of spies who have not yet been involved in proceedings. By induction we may reduce to the case where step $n - j + 1$ records the answer to a question asked to a spy. We may also assume that the candidate at this step is a spy, for otherwise step $n - j + 1$ is certainly downwards, and the inductive step is obvious. Let $-h$ be the height of the path after step $n - j$.

Suppose that step $n - j + 1$ is downwards, corresponding to the answer 'knight'. By induction we may assume that the spies continue to lie in their answers to the remaining questions. Hence assuming the downward step, the probability in the claim is the probability that a path starting at height $-(h+1)$ with k' upsteps and $s' - 1$ downsteps visits 0 from above exactly p times.

Now suppose that step $n - j + 1$ is upwards, corresponding to the answer 'spy'. As before, we may assume that from now on the spies always lie. Hence in this case, the probability in the claim is the probability that a path starting at height $-(h - 1)$ with k' upsteps and $s' - 1$ downsteps visits 0 from above exactly p times.

Shifting the paths upwards, we may apply Lemma 4.3 to show that the two probabilities are equal, *provided that* $h + 1 \leq k' - (s' - 1)$. But if $h > k' - s'$ and the remaining spies lie in all their answers, then the spy who is a candidate on step $n - j + 1$ will be accepted; we know this cannot happen. This proves the claim and hence the first part of the theorem.

By Lemma 4.4, the expected number of visits to 0 from above of a path with k upsteps and s downsteps is

$$\frac{1}{\binom{k+s}{s}} \sum_{i=0}^{s-1} \binom{k+s}{i}.$$

When spies lie in all their answers, this is the expected number of questions saved. We have just seen that the spies' answers when asked about one another do not affect the distribution of this quantity. This completes the proof of Theorem 4.1.

5. Open problems

It is natural to ask whether there is a questioning strategy which never uses more than $n + \ell - 1$ questions, and also has a reasonable probability of using fewer, no matter how cleverly the spies answer. By modifying the Spider Interrogation Strategy, the author has proved the following conjecture in the case where $g(\ell) > \ell^2$.

Conjecture 5.1. Let $g : \mathbf{N} \to \mathbf{N}$ be such that $g(\ell) > 2\ell$ for all $\ell \in \mathbf{N}$. Let $0 < \sigma \leq 1$. There is a questioning strategy which, provided that ℓ is sufficiently large, guarantees to use at most $g(\ell) + \ell - 1$ questions to find all the identities in an $g(\ell)$-person room known to contain at most ℓ spies, and in fact containing exactly $\lfloor \sigma\ell \rfloor$ spies, and will on average use at most $g(\ell) + 3\ell/4$ questions.

It seems likely that a more sophisticated questioning strategy, based on building long chains of people supporting one another, can be used to prove Conjecture 5.1 in the case when $g(\ell) = 2\ell + 1$. For more information about this strategy and the modified Spider Interrogation Strategy, the reader is referred to the supporting material on the author's website: `http://www.maths.bris.ac.uk/~mazmjw`.

The game-playing setting for Conjecture 5.1 is the variant form of 'Knights and Spies', in which the numbers of the spies are randomly chosen at the start of the game, and the Spy Master's only responsibility is to decide on their answers. The information that exactly $\lfloor \sigma\ell \rfloor$ spies are present is *not* revealed to the Interrogator, and need not by honoured by the Spy Master when refuting a claim.

We end with a final open problem which seems worthy of attention and may well be more tractable than Conjecture 5.1.

Problem 5.2. Let $1 \leq \ell < n/2$. In an n person room known to contain at most ℓ spies, what is the smallest number of questions that will guarantee to find at least one person's identity? What is the smallest number of questions that will guarantee to find a knight?

For example, given the sequence of questions shown in Figure 1, we can be sure after question 15 that person 15 is a knight (and also that person 18 is a spy), but before this question we cannot be certain of any single identity.

The Spider Interrogation Strategy shows that $2\ell - 1$ questions suffice to find a knight. This gives an upper bound for both parts of Problem 5.2. The author conjectures that this upper bound is in fact the correct answer to both parts. If so, we face the curious situation that, while we can find the identity of a particular person, nominated in advance, with 2ℓ questions, we can only save one question if the person is of our own choosing and to be nominated later.

To get a lower bound for either part of Problem 5.2 it seems natural to use the game-playing framework of §3. The Knight Hiding Strategy shows that ℓ questions are necessary, but it cannot otherwise be recommended, for if the Interrogator follows the Spider Interrogation Strategy, then he will find a knight after just ℓ questions. It is a reflection of the difficulty of Problem 5.2 that this variant form of 'Knights and Spies' seems harder to play than the original game. The author speculates that a better understanding of how to play this variant game might lead to some insights into how trust can be established in real-world social networks.

6. Appendix: Gamechecker

Practical experience suggests that when playing 'Knights and Spies', it is all too easy for the Spy Master inadvertently to answer in such a way that all consistent interpretations of his answers require strictly more than ℓ spies to be present; if this occurs the game must either be forfeited, or restarted from the point where the error occurred. Such errors can be flagged using the author's program *Gamechecker*, which makes an exhaustive search for an assignment of identities consistent with the Spy Master's responses. It reports if there is a unique such assignment, so it can also be used to adjudicate claims by the Interrogator. The Haskell source code for *Gamechecker* is available from the author's website: `http://www.maths.bris.ac.uk/~mazmjw`. It would be interesting to know whether there is a polynomial-time algorithm for deciding whether an incomplete game is in a consistent state; the back-tracking algorithm used by *Gamechecker* works well in practice, but in the worst case requires exponential time.

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