DIAGONAL SUMS OF GENERALIZED PASCAL TRIANGLES

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1. INTRODUCTION

A sequence of generalized Fibonacci numbers $u(n; p,q)$ which can be interpreted as sums along diagonals in Pascal's triangle appear in papers by Harris and Styles [1], [2]. In this paper, Pascal's binomial coefficient triangle is generalized to trinomial and other polynomial coefficient arrays, and a method is given for finding the sum of terms along any rising diagonals in any such array, given by $(1 + x + \cdots + x^{r-1})^n$, $n = 0, 1, 2, 3, \cdots$, $r \geq 2$.

2. THE TRINOMIAL TRIANGLE

If we write only the coefficients appearing in expansions of the trinomial, $(1 + x + x^2)^n$, we are led to the following array:

1												
1	1	1										
1	2	3	3	1								
1	3	6	7	6	3	1						
1	4	10	16	19	16	10	4	1				
1	5	15	30	45	51	45	30	15	5	1		
1	6	21	50	90	126	141	126	90	50	21	6	1

...

Call the top row the zeroth row and the left column the zeroth column. Then, let

$$G_0 = \frac{1}{1 - x}, \qquad G_1 = \frac{x}{(1 - x)^2}, \qquad G_2 = \frac{x}{(1 - x)^3}$$

be the column generators as the columns are positioned above. The general recurrence for the column generators is

(1) $$G_{n+2} = \frac{x}{1-x}(G_{n+1} + G_n) .$$

Let

$$G = \sum_{n=0}^{\infty} G_n = \sum_{n=0}^{\infty} u(n;\ 0,1)x^n .$$

The sum G in the general case will have for the coefficient of x^n the number $u(n;\ p,q)$, which, as applied to the trinomial triangle, will be the sum of the term in the left column and the n^{th} row and the terms obtained by taking steps p units up and q units to the right. That is, $u(n;\ p,q)$ is a member of a sequence of sums whose terms lie on particular diagonals of the trinomial triangle. To find G, for $p = 0$ and $q = 1$, we use the method of Polya [3] and the recurrence relation (1). Let S_n be the sum of the first n terms of G.

$$G_2 = \frac{x}{1-x}(G_1 + G_0)$$

$$G_3 = \frac{x}{1-x}(G_2 + G_1)$$

$$\cdots$$

$$G_{n+1} = \frac{x}{1-x}(G_n + G_{n-1})$$

$$G_{n+2} = \frac{x}{1-x}(G_{n+1} + G_n) .$$

Summing vertically,

$$S_n + G_{n+2} + G_{n+1} - G_0 - G_1 = \frac{x}{1-x}(S_n + G_{n+1} - G_0 + S_n)$$

$$S_n\left(1 - \frac{2x}{1-x}\right) = G_0\left(1 - \frac{x}{1-x}\right) + G_1 + G_{n+1}\left(\frac{x}{1-x} - 1\right) - G_{n+2} .$$

It can be shown that $\lim_{n\to\infty} G_n = 0$ for $|x| < 1/r$, $r > 2$, so that

$$G = \lim_{n\to\infty} S_n = \frac{1-x}{1-3x}\left(\frac{1-2x}{(1-x)^2} + \frac{x}{(1-x)^2}\right) = \frac{1}{1-3x},$$

which was to be expected, since

$$\frac{1}{1-3x} = 1 + 3x + 9x^2 + 27x^3 + 81x^4 + 245x^5 + \cdots ,$$

where each coefficient is the sum of an appropriate row in the triangle. In fact, each coefficient of x^n is $u(n; 0,1) = 3^n$.

Now, let us consider $u(n; p,1)$. Here

$$G_0^\star = 1/(1-x) \quad \text{and} \quad G_1^\star = x^{p+1}/(1-x)^2$$

with recurrence

$$G_{n+2}^\star = \frac{x}{1-x}(x^p G_{n+1}^\star + x^{2p} G_n^\star) . \tag{2}$$

(Notice that multiplication by x^p and x^{2p} allows for moving up p rows in the triangle.) Following Polya's method of summing vertically as before,

$$S_n^\star\left(1 - \frac{x^{p+1} + x^{2p+1}}{1-x}\right) = \frac{1}{1-x}\left(1 - \frac{x^{p+1}}{1-x}\right) + \frac{x^{p+1}}{(1-x)^2} + \left(\frac{x^{p+1}}{1-x} - 1\right)G_{n+1}^\star - G_{n+2}^\star .$$

Since, again, $\lim_{n\to\infty} G_n^\star = 0$, for $|x| < 1/r$, $r > 2$, so that

$$G^\star = \lim_{n\to\infty} S_n^\star = \frac{1}{1 - x - x^{p+1} - x^{2p+1}} .$$

Now, if $p = 1$, we get the generating function for the Tribonacci numbers, $G = 1/(1 - x - x^2 - x^3)$. The Tribonacci numbers T_n (see [4]) are 1, 1, 2, 4, 7, 13, 24, $\cdots$, where each term after the third is the sum of the preceding three terms. That is, $u(n;\ 1,1) = T_{n+1}$. For a particular verification, the reader is invited in each case to perform the indicated division.

Now, if we let $q = 2$, then we must deal with every other term of the column generator recurrence relation. To solve $u(n;\ 0,2)$, $G_0 = 1/(1 - x)$, $G_2 = x/(1 - x)^3$, and the recurrence (1) originally considered, leads to

$$(3) \qquad G_{2n+4} = \left(\frac{x^2}{(1 - x)^2} + \frac{2x}{1 - x}\right) G_{2n+2} - \frac{x^2}{(1 - x)^2} \cdot G_{2n} \ .$$

Following the same method as before, we have, for

$$S_n = \sum_{i=0}^{n} G_{2i} \ ,$$

$$S_n \left(1 - \frac{2x - x^2}{(1 - x)^2} + \frac{x^2}{(1 - x)^2}\right) = G_0 \left(1 - \frac{2x - x^2}{(1 - x)^2}\right) + G_2 + R_n \ ,$$

where R_n is a term involving G_{2n+4} and G_{2n+2}. Again, since

$$\lim_{n\to\infty} G_n = 0, \quad \lim_{n\to\infty} R_n = 0, \quad |x| < 1/r, \quad r > 2,$$

$$\lim_{n\to\infty} S_n = G = \frac{1 - 2x}{1 - 4x + 3x^2} = \sum_{n=0}^{\infty} u(n;\ 0,2)x^n \ .$$

This gives us a generating function for sums of alternate terms of rows in the trinomial triangle.

Let $p = 1$ and $q = 2$ and return to $G_0 = 1/(1-x)$, $G_2 = x^2/(1 - x)^2$, and, from recurrence (3),

$$G^{\star}_{2n+4} = x\left(\frac{x^2}{(1 - x)^2} + \frac{2x}{1 - x}\right) G^{\star}_{2n+2} - \frac{x^2 \cdot x^2}{(1 - x)^2} G^{\star}_{2n} \ ,$$

where we must multiply by x and x^2 to account for moving up one row through the trinomial array. Going to the Polya method again to find $u(n;\ 1,2)$ we have, for

$$S_n^\star = \sum_{i=0}^{n} G_{2i}^\star ,$$

$$S_n^\star\left(1 - \frac{(2x - x^2)x}{(1 - x)^2} + \frac{x^2 \cdot x^2}{(1 - x)^2}\right) = G_0^\star\left(1 - \frac{x(2x - x^2)}{(1 - x)^2}\right) + G_2^\star + R_n ,$$

where R_n involves only terms $G_{2n+2}^\star$ and $G_{2n+4}^\star$, so that $\lim_{n\to\infty} R_n = 0$, $|x| < 1/r,\ r > 2$.

$$S_n^\star\left(\frac{1 - 2x - x^2 + x^3 + x^4}{(1 - x)^2}\right) = (1 - 2x + x^2 - 2x^2 + x^3 + x^2)/(1 - x)^3 + R_n$$

$$G^\star = \lim_{n\to\infty} S_n^\star = \frac{1 - 2x + x^3}{(1 - x)(1 - 2x - x^2 + x^3 + x^4)} ,$$

which simplifies to

$$\frac{1 - x - x^2}{1 - 2x - x^2 + x^3 + x^4} = \sum_{n=0}^{\infty} u(n;\ 1,2)x^n .$$

Returning now to the more general case, we find the generating function for the numbers $u(n;\ p,2)$. Using the recurrence relation (3), but allowing for moving up p rows in the triangle, and then summing vertically as before yields

$$S_n\left(1 - \frac{(2x - x^2)x^p}{(1 - x)^2} + \frac{x^2 \cdot x^{2p}}{(1 - x)^2}\right) = \frac{1}{1 - x}\left(1 - \frac{x^p(2x - x^2)}{(1 - x)^2}\right)$$

$$+ \frac{x^{p+1}}{(1 - x)^3} + R_n ,$$

where again $\lim_{n\to\infty} R_n = 0$ for $|x| < 1/r$. Simplifying the above, and letting $\lim_{n\to\infty} S_n = G$,

$$G = \frac{(1 - 2x + x^2 + x^{p+2} - x^{p+1})/(1 - x)}{(1 - x)^2 - 2x^{p+1} + x^{p+2} + x^{2p+2}}$$

$$= \frac{1 - x - x^{p+1}}{(1 - x)^2 - 2x^{p+1} + x^{p+2} + x^{2p+2}} = \sum_{n=0}^{\infty} u(n;\ p, 2)\, x^n .$$

This agrees with the previous cases for $p = 1$, $q = 2$ and for $p = 0$, $p = 2$.

In seeking the numbers $u(n;\ 0, 3)$, we need the recurrence relation

$$G_{3(n+2)} = \frac{3x^2 - 2x^3}{(1 - x)^3} G_{3(n+1)} + \frac{x^3}{(1 - x)^3} G_{3n} ,$$

which, following the previous method, gives

$$S_n\left(1 - \frac{3x^2 - 2x^3}{(1 - x)^3} - \frac{x^3}{(1 - x)^3}\right) = \frac{1}{1 - x}\left(1 - \frac{3x^2 - 2x^3}{(1 - x)^3}\right)$$

$$+ \frac{2x^2 - x^3}{(1 - x)^4} + R_n ,$$

and

$$\sum_{n=0}^{\infty} u(n;\ 0, 3)x^n = \frac{1 - 2x}{1 - 3x} = 1 + \sum_{n=0}^{\infty} 3^n x^{n+1} .$$

In fact,

$$\sum_{n=0}^{\infty} u(n;\ p, 3)x^n = \frac{1 - 2x + x^2 - x^{p+2}}{(1 - x)^3 - 3x^{p+2} + 2x^{p+3} - x^{2p+3}} .$$

3. QUADRINOMIALS, PENTANOMIALS, AND HEXANOMIALS

If we consider the array of coefficients which arise in the expansion of the quadrinomial $(1 + x + x^2 + x^3)^n$,

1												
1	1	1	1									
1	2	3	4	3	2	1						
1	3	6	10	12	12	10	6	3	1			
1	4	10	20	31	40	44	40	31	20	10	4	1

$$\cdots$$

and use the methods of the preceding section, the expressions given below can be derived without undue difficulty. For the quadrinomial coefficients, the generating functions are given by

$$G_{n+3} = \frac{x}{1-x}(G_{n+2} + G_{n+1} + G_n)$$

where

$$G_0 = 1/(1-x), \quad G_1 = x/(1-x)^2, \quad G_2 = x/(1-x)^3, \quad G_3 = x/(1-x)^4,$$
$$G_4 = (3x^2 - 3x^3 + x^4)/(1-x)^5 \quad .$$

It is easy to find that $u(n;\,0,1) = 4^n$ and $u(n;\,1,1) = Q_{n+1}$, where Q_n is the quadrinacci number given by $1, 1, 2, 4, 8, 15, 29, \cdots$, where each term after the fourth is the sum of the preceding four terms (see [4]). The generating function for Q_n is $1/(1 - x - x^2 - x^3 - x^4)$, and

$$\sum_{n=0}^{\infty} u(n;\,p,1)x^n = \frac{1}{1 - x - x^{p+1} - x^{2p+1} - x^{3p+1}} \quad .$$

From the recursion

$$G_{2(n+3)} = \frac{2x - x^2}{(1-x)^2} G_{2(n+2)} + \frac{x^2}{(1-x)^2} G_{2(n+1)} + \frac{x^2}{(1-x)^2} G_{2n} \ ,$$

one finds

$$\sum_{n=0}^{\infty} u(n;\ 0,2)x^n = \frac{1-2x}{1-4x} = 1 + \sum_{n=1}^{\infty} 2^{2n-1} x^n ;$$

$$\sum_{n=0}^{\infty} u(n;\ 1,2)\ x^n = \frac{1 - x - x^2}{1 - 2x - x^2 + x^3 - x^4 - x^5} ;$$

$$\sum_{n=0}^{\infty} u(n;\ p,2)\ x^n = \frac{1 - x - x^{p+1}}{(1-x)^2 - 2x^{p+1} + x^{p+2} - x^{2p+2} - x^{3p+2}} .$$

Also, from

$$G_{3(n+3)} = \frac{3x - 3x^2 + x^3}{(1-x)^3} G_{3(n+2)} - \frac{3x^2 - x^3}{(1-x)^3} G_{3(n+1)} + \frac{x^3}{(1-x)^3} G_{3n} ,$$

one finds

$$\sum_{n=0}^{\infty} u(n;\ 0,3)x^n = \frac{1 - 3x}{1 - 5x + 4x^2}$$

If one continues in a similar way, the analogous results for the pentanomial becomes $u(n;\ 0,1) = 5^n$;

$$\sum_{n=0}^{\infty} u(n;\ p,1)\ x^n = \frac{1}{1 - x - x^{p+1} - x^{2p+1} - x^{3p+1} - x^{4p+1}}$$

where $u(n;\ 1,1) = 1, 1, 2, 4, 8, 16, 31, 61, \cdots,$ and each term after the fifth is the sum of the preceding five terms:

$$\sum_{n=0}^{\infty} u(n;\ 0,2)x^n = \frac{1 - 3x}{1 - 6x + 5x^2} ;$$

$$\sum_{n=0}^{\infty} u(n;\ p,2)x^n = \frac{1 - x - x^{p+1} - x^{2p+1}}{(1-x)^2 - 2x^{p+1} + x^{p+2} - 2x^{2p+1} + x^{2p+2} + x^{3p+2} + x^{4p+2}} .$$

For the hexanomial, we can derive $u(n;\ 0,1) = 6^n$;

$$\sum_{n=0}^{\infty} u(n;\ p,1)\, x^n = \frac{1}{1 - x - x^{p+1} - x^{2p+1} - x^{3p+1} - x^{4p+1} - x^{5p+1}};$$

$$\sum_{n=0}^{\infty} u(n;\ 0,2)x^n = \frac{1 - 3x}{1 - 6x};$$

$$\sum_{n=0}^{\infty} u(n;\ p,2)x^n = \frac{1 - x - x^{p+1} - x^{2p+1}}{(1-x)^2 - 2x^{p+1} + x^{p+2} - 2x^{2p+1} + x^{2p+2} - x^{3p+2} - x^{4p+2} - x^{5p+2}}$$

In general, for a k-nomial (k terms) coefficient array, one discovers that $u(n;\ 0,1) = k^n$ and $u(n;\ 0,k) = k^{n-1}$, $n \geq 1$. Now we can readily generalize our results.

4. GENERALIZATION OF TRINOMIAL CASE

In the quadratic equation $y^2 - ay + b = 0$, let $a = b = x/(1 - x)$. Then, if r_1 and r_2 are the roots of the above quadratic, let

$$r_1^k + r_2^k = P_k\left(\frac{x}{1 - x}, \frac{x}{1 - x}\right),$$

given by $P_0 = 2$, $P_1 = x/(1 - x)$,

$$P_2 = \left(\frac{x}{1 - x}\right)^2 + \frac{2x}{1 - x},$$

and satisfying

$$P_{k+2} = \frac{x}{1 - x}(P_{k+1} + P_k).$$

Now, the recurrence relation for the column generators for the trinomial case is (let $q = k$)

$$G_{(n+2)k} = P_k G_{(n+1)k} + (-1)^{k+1}\left(\frac{x}{1 - x}\right)^k G_{nk},$$

leading to

$$G^{\star}_{(n+2)k} = x^p P_k G^{\star}_{(n+1)k} + \frac{(-1)^{k+1} x^{2p} x^k}{(1 - x)^k} G^{\star}_{nk}$$

where $G^{\star}_{nk} = x^{np} G_{nk}$ to allow for moving p steps up through the triangle. Then, summing vertically gives

$$S_n \left(1 - P_k x^p + \frac{(-1)^k x^{2p+k}}{(1 - x)^k} \right) = G^{\star}_0 (1 - P_k x^p) + G^{\star}_k + R_n \,,$$

where $\lim_{n \to \infty} R_n = 0$, $|x| < 1/r$, $r > 2$.

Hence,

$$G \left(\frac{(1 - x)^k - x^p P_k (1 - x)^k + (-1)^k x^{2p+k}}{(1 - x)^k} \right) = \frac{1}{1 - x} (1 - P_k x^p) + x^p G_k$$

for the column generators defined in Eq. (1).

Applying the formula given by Bicknell and Draim [5],

$$P_k = \sum_{i=0}^{[k/2]} \frac{k(k - i - 1)!}{(k - 2i)!\, i!} \cdot \left(\frac{x}{1 - x} \right)^{k-i} \,,$$

$[x]$ the greatest integer function, gives an explicit formula for G. Since G is the generating function for the numbers $u(n; p, k)$, we have resolved our problem for the trinomial triangle. Harris and Styles [1] have solved the binomial case by summing diagonals of Pascal's triangle. Feinberg in [6] has given series convergents for $u(n; p, 1)$ for the trinomial and quadrinomial cases. We now move on to the solution of the general case for the array of coefficients formed from polynomials of n terms.

5. SYMMETRIC FUNCTIONS AND COLUMN GENERATORS: THE GENERAL CASE

Let

$$P(x) = x^n - p_1x^{n-1} + p_2x^{n-2} - \cdots + (-1)^j p_j x^{n-j} + \cdots + (-1)^n p_n ,$$

where p_j is the j^{th} symmetric function of the roots of $P(x) = 0$. (For a discussion of symmetric functions, see [7] and [8].) Now let $p_j(k)$ be the j^{th} symmetric function of the k^{th} powers of the roots of $P(x)$. Then

$$p_1(m + n) - p_1(m + n - 1)p_1 + p_1(m + n - 2)p_2 - \cdots + (-1)^n p_1(m)p_n \equiv 0 ,$$

since each p_i represents sums of the products of solutions which are geometric progression solutions to the original difference equation whose auxiliaty polynomial is listed above. Thus we need n starting values for each such sequence.

If

$$G_{n+2} = \frac{x}{1 - x} (G_{n+1} + G_n) ,$$

then

$$G_{(n+2)q} = p_1(q)G_{(n+1)q} + (-1)^{q+1}\left(\frac{x}{1 - x}\right)^q G_{nq} ,$$

where

$$p_1(0) = 2, \quad p_1(1) = x/(1 - x);$$

and

$$p_1(m + 2) = \frac{x}{1 - x} (p_1(m + 1) + p_1(m)) ,$$

with auxiliary polynomial

$$y^2 - \frac{x}{1-x}y - \frac{x}{1-x} \quad .$$

This is the resolution of our trinomial case, expressed in a modified form.

The column generators for the quadrinomial case will be related by

$$G_{n+3} = \frac{x}{1-x}(G_{n+2} + G_{n+1} + G_n)$$

where

$$G_{(n+3)q} = p_1(q)G_{(n+2)q} - p_2(q)G_{(n+1)q} + p_3(q)G_{nq} \,.$$

Here, the auxiliary polynomial is

$$y^3 - p_1(1)y^2 + p_2(1)y - p_3(1)$$

where

$$p_1(1) = p_3(1) = -p_2(1) = \frac{x}{1-x} \,.$$

Now,

$$p_2(k) = (p_1^2(k) - p_1(2k))/2$$

$$p_3(k) = \left(\frac{x}{1-x}\right)^k \quad .$$

Next,

$$p_1(0) = 3, \quad p_1(1) = x/(1-x), \quad p_1(2) = \left(\frac{x}{1-x}\right)^2 + \frac{2x}{1-x} \,,$$

and

$$p_1(m+3) = \frac{x}{1-x}(p_1(m+2) + p_1(m+1) + p_1(m)) \,.$$

Notice that, since our values for $p_1(q)$, $p_2(q)$, and $p_j(q)$ are defined sequentially and since moving up p rows can be adjusted by multiplying by x^p, we can solve the quadrinomial case. To derive $u(n; p,q)$, we can use $(G^\star_{iq} = x^{ip}G_q)$

$$G^\star_{(n+3)q} = x^p p_1(q) G_{(n+2)q} - x^{2p} p_2(q) G_{(n+1)q} + x^{3p} p_3(q) G^\star_{nq} ,$$

leading to

$$S_n(1 - x^p p_1(q) + x^{2p} p_2(q) - x^{3p} p_3(q)) = G_0(1 - x^p p_1(q) + x^{2p} p_2(q))$$
$$+ x^{2p} G_q(1 - x^p p_1(q)) + x^{4p} G_{2q} + R_n ,$$

where $\lim_{n \to \infty} R_n = 0$, $|x| < 1/r$, $r > 2$.

Using formulas given by Bicknell and Draim [9],

$$p_1(q) = \sum_{k=0}^{[q/3]} \sum_{n=0}^{\left[\frac{q-3k}{2}\right]} \frac{q(q - m - 2k - 1)!}{(q - 2n - 3k)!\, m!\, k!} \cdot \left(\frac{x}{1 - x}\right)^{q-m-2k} ,$$

$$p_2(q) = \sum_{n=0}^{q/3} \sum_{n=0}^{\left[\frac{q-3k}{2}\right]} \frac{q(q - m - 2k - 1)!}{(q - 2n - 3k)!\, m!\, k!} \cdot \left(\frac{x}{1 - x}\right)^{q-k} (-1)^{q-m-3k} ,$$

$$p_3(q) = \left(\frac{x}{1 - x}\right)^q , \quad [x] \text{ the greatest integer function,}$$

we actually could write an explicit formula for G, the generating function for the numbers $u(n; p,q)$ for the quadrinomial case.

For the pentanomial case, we would go to

$$G_{n+4} = \frac{x}{1 - x}(G_{n+3} + G_{n+2} + G_{n+1} + G_n) ,$$

with auxiliary polynomial

$$y^4 - p_1(1)y^3 + p_2(1)y^2 - p_3(1)y + p_4(1) = 0 ,$$

where

$$p_1(1) = p_3(1) = -p_2(1) = -p_4(1) = x/(1 - x) .$$

Then we need

$$p_1(0) = 4, \quad p_1(1) = x/(1 - x), \quad p_1(2) = (2x - x^2)/(1 - x)^2,$$
$$p_1(3) = (3x - 3x^2 + x^3)/(1 - x)^3,$$

and

$$p_1(n + 4) = \frac{x}{1 - x}\,(p_1(n + 3) + p_1(n + 2) + p_1(n + 1) + p_1(n));$$

$$p_2(k) = (p_1^2(k) - p_1(2k))/2 ,$$

$$p_3(k) = (p_1^3(k) - 3p_1(2k)p_1(k) + 2p_1(3k))/6, \text{ (see [7])}$$

$$p_4(k) = (-1)^k\left(\frac{x}{1 - x}\right)^k .$$

The relationship

$$G^\star_{(n+4)q} = x^p p_1(q)G^\star_{(n+3)q} - x^{2p}p_2(q)G_{(n+2)q} + x^{3p}p_3(q)G_{(n+1)q} - x^{4p}p_4(q)G_{nq} ,$$

$G^\star_{iq} = x^{ip}G_{iq}$, combined with our earlier techniques provides a general solution for $u(n; p,q)$ for the pentanomial case, although it would be a messy computation. However, if one notes some of the relationships between the $p_1(k)$ for the polynomials

$$y^{n-1} - \frac{x}{1 - x}\,(y^{n-2} + y^{n-3} + \cdots + y + 1) = 0$$

for different values of n, much of the labor is taken out of the computation. The expressions $p_1(k)$ are identical for the polynomial with n terms and the polynomial with (n - 1) terms for $k = 1, 2, 3, \cdots, n - 2$; $p_1(0) = n$ for all cases; and

$$P_1(m + n - 1) = \frac{x}{1 - x}\left(\sum_{i=2}^{n} p_1(m + m - i)\right)$$

In fact,

$$p_1(k) = \frac{1}{(1 - x)^k} - 1$$

for $k = 1, 2, \cdots, n - 1$ for the polynomial with n terms. Thus, $p_1(k)$ can be derived sequentially for any value of k for the polynomial with n terms given by

$$y^{n-1} - \frac{x}{1 - x}(y^{n-2} + \cdots + y + 1) = 0 .$$

We can sequentially generate all sums of powers of the roots of any polynomial because we can get the proper starting values sequentially as well as find higher powers sequentially.

Now, it is well known that, given all the sums of the powers of the roots, $p_1(0)$, $p_1(1)$, $p_1(2)$, $\cdots$, $p_1(n)$, for a given fixed polynomial, one can determine the other symmetric functions of the roots in terms of the $p_1(k)$. (See [7], [8].) Waring's formula gives

$$p_m(k) = (-1)^r \cdot \frac{(p_1(k))^{r_1} \cdot (p_1(2k))^{r_2} \cdot (p_1(3k))^{r_3} \cdots (p_1(mk))^{r_m}}{(r_1!\, r_2!\, r_3! \cdots r_m!)(1^{r_1} \cdot 2^{r_2} \cdot 3^{r_3} \cdots m^{r_m})}$$

$$r_1 + r_2 + r_3 + \cdots + r_m = r$$

$$r_1 + 2r_2 + 3r_3 + \cdots + mr_m = m .$$

Also, the generating functions for the coefficients arising in the expansion of the n-nomial $(1 + x + x^2 + \cdots + x^{n-1})^k$ can be derived sequentially by $G_1 = x^i/(1-x)^{i+1}$, $i = 0,1,2,\cdots, n-1$, $G_n = ((1-x)^{n-1} - 1)/(1-x)^{n+1}$, and $G_{n+1} = x/(1-x) \cdot (G_n + G_{n-1} + \cdots + G_2 + G_1 + G_0)$. Thus, for the polynomial with n terms, by taking $G^\star_{iq} = x^{ip}G_{iq}$, letting

$$G^\star_{(m+n-1)q} = \sum_{i=1}^{n-1} (-1)^{i+1} p_i(q) G^\star_{(m+n-1-i)} ,$$

and using the methods of this paper, the generating function for the numbers $u(n;\, p,q)$ could be derived.

In [11] it was promised a proof that, for $p = 1$,

$$\sum_{n=0}^{\infty} u(n;p,1)x^n = \frac{1}{1 - x - x^{p+1} - x^{2p+1} - \cdots - x^{(r-1)p+1}}$$

for the general r-nomial triangle induced by the expansion

$$(1 + x + x^2 + \cdots + x^{r-1})^n \qquad n = 0,1,2,3,\cdots .$$

This follows from the definition. Let the r-nomial triangle be left justified and take sums by starting on the left edge and jumping up p and over 1 entry repeatedly until out of the triangle. Thus,

$$u(n;p,1) = \sum_{k=0}^{\left[\frac{n(r-1)}{p+1}\right]} \left\{ {n - kp \atop k} \right\}_r ,$$

where

$$(1 + x + x^2 + \cdots + x^{r-1})^n = \sum_{j=0}^{n(r-1)} \left\{ {n \atop j} \right\}_r x^j .$$

The r-nomial coefficient $\left\{ {n \atop j} \right\}_r$ is the entry in the n^{th} row and j^{th} column of the generalized Pascal triangle. Thus

$$\frac{1}{1 - x(1 + x^p + x^{2p} + \cdots + x^{p(r-1)})} = \sum_{n=0}^{\infty} \left[x(1 + x^p + x^{2p} + \cdots + x^{p(r-1)})\right]^n$$

$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\left[\frac{n(r-1)}{p+1}\right]} \left\{ {n - kp \atop k} \right\}_r \right) x^n .$$

(Continued on p. 393.)

TABLE OF GENERATING FUNCTIONS FOR POLYNOMIAL TRIANGLE DIAGONAL SUMS

u(n; p,q)	Binomial [1]	Trinomial	Quadrinomial
u(n; 0,1)	$\frac{1}{1 - 2x}$	$\frac{1}{1 - 3x}$	$\frac{1}{1 - 4x}$
u(n; 1,1)	$\frac{1}{1 - x - x^2}$	$\frac{1}{1 - x - x^2 - x^3}$	$\frac{1}{1 - x - x^2 - x^3 - x^4}$
u(n; p,1)	$\frac{1}{1 - x - x^{p+1}}$	$\frac{1}{1 - x - x^{p+1} - x^{2p+1}}$	$\frac{1}{1 - x - x^{p+1} - x^{2p+1} - x^{3p+1}}$
u(n; 0,2)	$\frac{1 - x}{1 - 2x}$	$\frac{1 - 2x}{1 - 4x + 3x^2}$	$\frac{1 - 2x}{1 - 4x}$
u(n; 1,2)	$\frac{1 - x}{1 - 2x + x^2 - x^3}$	$\frac{1 - x - x^2}{1 - 2x - x^2 + x^3 + x^4}$	$\frac{1 - x - x^2}{1 - 2x - x^2 + x^3 - x^4 - x^5}$
u(n; p,2)	$\frac{1 - x}{(1 - x)^2 - x^{p+2}}$	$\frac{1 - x - x^{p+1}}{(1 - x)^2 - 2x^{p+1} + x^{p+2} = x^{2p+2}}$	$\frac{1 - x - x^{p+1}}{(1 - x)^2 - 2x^{p+1} + x^{p+2} - x^{2p+2} - x^{3p+2}}$
u(n; 0,3)	$\frac{1 - 2x + x^2}{1 - 3x + 3x^2 - 2x^3}$	$\frac{1 - 2x}{1 - 3x}$	$\frac{1 - 3x}{1 - 5x + 4x^2}$
u(n; p,3)	$\frac{(1 - x)^2}{(1 - x)^3 - x^{p+3}}$	$\frac{(1 - x)^2 - x^{p+2}}{(1 - x)^3 - 3x^{p+2} + 2x^{p+3} - x^{2p+3}}$	
u(n; 0,4)	$\frac{1 - 3x + 3x^2 - x^3}{1 - 4x + 6x^2 - 4x^3}$	$\frac{1 - 3x + 2x^2 - x^3}{1 - 4x + 4x^2 - 4x^3 + 3x^4}$	$\frac{1 - 3x}{1 - 4x}$
u(n; p,q)	$\frac{(1 - x)^{q-1}}{(1 - x)^q - x^{p+q}}$		

TABLE OF GENERATING FUNCTIONS FOR POLYNOMIAL TRIANGLE DIAGONAL SUMS

u(n; p,q)	Pentanomial	Hexanomial
u(n; 0,1)	$\frac{1}{1 - 5x}$	$\frac{1}{1 - 6x}$
u(n; 1,1)	$\frac{1}{1 - x - x^2 - x^3 - x^4 - x^5}$	$\frac{1}{1 - x - x^2 - x^3 - x^4 - x^5 - x^6}$
u(n; p,1)	$\frac{1}{1 - x - x^{p+1} - x^{2p+1} - x^{3p+1} - x^{4p+1}}$	$\frac{1}{1 - x - x^{p+1} - x^{2p+1} - x^{3p+1} - x^{4p+1} - x^{5p+1}}$
u(n; 0,2)	$\frac{1 - 3x}{1 - 6x + 5x^2}$	$\frac{1 - 3x}{1 - 6x}$
u(n; 1,2)	$\frac{1 - x - x^2 - x^3}{1 - 2x - x^2 - x^3 + x^4 + x^5 + x^6}$	$\frac{1 - x - x^2 - x^3}{1 - 2x - x^2 - x^3 + x^4 - x^5 - x^6 - x^7}$
u(n; p,2)	$\frac{1 - x - x^{p+1} - x^{2p+1}}{(1-x)^2 - 2x^{p+1} + x^{p+2} - 2x^{2p+1} + x^{2p+2} + x^{3p+2} + x^{4p+2}}$	$\frac{1 - x - x^{p+1} - x^{2p+1}}{(1-x)^2 - 2x^{p+1} + x^{p+2} - 2x^{2p+1} + x^{2p+2} - x^{3p+2} - x^{4p+2} - x^{5p+2}}$
u(n; 0,3)	$\frac{1 - 4x + 2x^2}{1 - 6x + 6x^2 - 5x^3}$	$\frac{1 - 4x}{1 - 6x}$

r-nomial

u(n;p,1) $\qquad \frac{1}{1 - x - x^{p+1} - x^{2p+1} - \dots - x^{(r-1)p+1}}$