RENCONTRES GRAPHS: A FAMILY OF BIPARTITE GRAPHS*

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I. INTRODUCTION

A number of different families of graphs have recently been proposed as possible interconnection models for computer networks. A tree is the cheapest interconnection, but has unacceptably poor connectivity properties. On the other hand, the complete graphs K_n, although most reliable and best connected, is prohibitively expensive (too many edges). A number of other graph families that lie between these two extremes have been proposed and analyzed for relevant properties such as path lengths, connectivities, cost, reliability, potential congestions, throughput, etc. The search for "good" interconnection graphs for various situations continues. This paper is an outcome of our attempt to find a class of graphs which satisfy certain desired properties.

In Section II, we derive a family of adjacency matrices from Rencontres numbers, and call the corresponding graphs Rencontres graphs, which are connected, undirected, bipartite graphs. In Section III, the connectivity of Rencontres graphs is explored. In that section, we also prove that the complete bipartite graph $K_{t,t}$ is a subgraph of the Rencontres graph of 2^t vertices. An expression for the number of edges in a Rencontres graph in terms of the number of vertices is developed in Section IV. In Section V, it is shown that all Rencontres matrices of order other than 2 are singular.

We have used standard graph theoretic terms, for which readers may refer to [3] or [4]. All logarithms are with respect to base 2.

II. BASIC CONCEPTS AND DEFINITIONS

A classical combinatorial problem, known generally by its French name, "le problème des rencontres," is to find the number of permutations of n distinct elements (say, 1, 2, ..., n) such that no element is in its own position, or element k is not in the k^{th} position, $k = 1, 2, \ldots, n$. It is also known as the derangement problem. Its solution by Montmort (1713) effectively uses the principle of inclusion and exclusion [1]. More generally, the derangement problem enumerates permutations of n distinct elements according to the number of elements in "their own positions."

Let $D_{n,k}$ be the number of permutations of n elements with exactly k of them not displaced. In particular, $D_{n,0}$ is the number of permutations of n elements with all of them displaced, and $D_{n,n}$ is the number of permutations of n elements with none of them displaced. It has been shown in [1] that

$$D_{n,k} = \binom{n}{k} D_{n-k,0}.$$

The numbers $D_{n,k}$ for given n and k, $0 \leqslant k \leqslant n$, are called *Rencontres numbers*.

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For $n = 0, 1, \ldots, 10$ and $k = 0, 1, \ldots, 10$, the numbers $D_{n,k}$ are given in Table 1, henceforth referred to as the *Rencontres table*.

Table 1. Rencontres Numbers $D_{n,k}$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	0	1									
2	1	0	1								
3	2	3	0	1							
4	9	8	6	0	1						
5	44	45	20	10	0	1					
6	265	264	135	40	15	0	1				
7	1854	1855	924	315	70	21	0	1			
8	14833	14832	7420	2464	630	112	28	0	1		
9	133496	133497	66744	22260	5544	1134	168	36	0	1	
10	1334961	1334960	667485	222480	55650	11088	1890	240	45	0	1

The following results can be derived easily.

$$D_{0,0} = 1$$

$$D_{n,n} = \binom{n}{n} D_{0,0} = 1 \text{ for all } n \geqslant 0$$

$$D_{n,0} = nD_{n-1,0} + (-1)^n \text{ for all } n \geqslant 1$$

$$D_{n+1,n} = 0 \text{ for all } n \geqslant 0$$

$$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k,0} \text{ for all } n \geqslant 0$$

$$D_{n,k} = D_{n-1,k-1} + \binom{n-1}{k} D_{n-k,0} \text{ for all } n \geqslant 1 \text{ and } 1 \leqslant k \leqslant n$$

$$D_{i,j} = 0 \text{ if either or both } i \text{ and } j \text{ are negative integers.}$$

Let us define a few terms used in this paper.

Definition 1: An $n \times n$ symmetric binary matrix is called the *Rencontres matrix* $RM(n)$ of order n if its principal diagonal entries are all 0's and its lower triangle (and therefore the upper also) consists of the first $n - 1$ rows of the Rencontres table modulo 2. Let $rm_{i,j}$ denote the element in the i^{th} row and the j^{th} column of the Rencontres matrix.

Definition 2: The simple, undirected graph with n vertices corresponding to $RM(n)$ as its adjacency matrix is called the *Rencontres graph* $RG(n)$ of order n.

The matrix $RM(10)$ is shown below followed (in Figure 1) by the first eight Rencontres graphs.

$$
\begin{array}{c}
\\
1\\ 2\\ 3\\ 4\\ 5\\ 6\\ 7\\ 8\\ 9\\ 10
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}\\
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0\\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

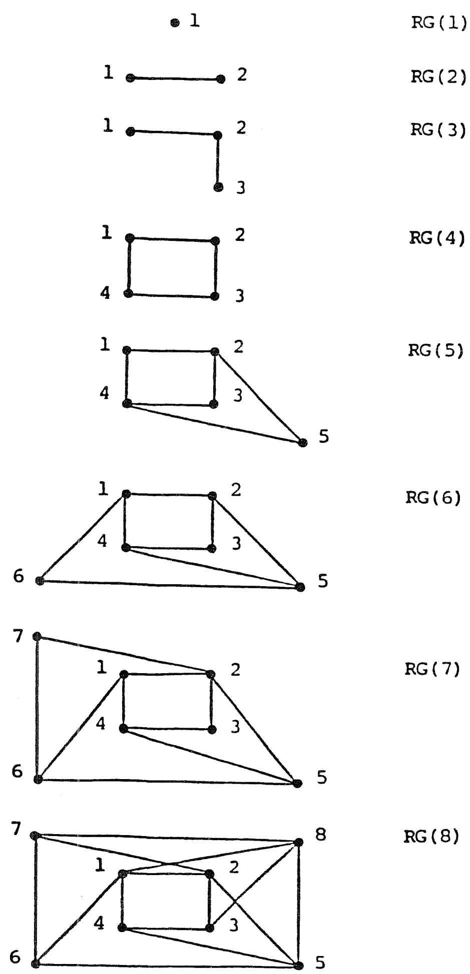

Figure 1. Rencontres Graphs $RG(n)$, $1 \leqslant n \leqslant 8$

Definition 3: Let $rt_{i,j}$ be the j^{th} element in the i^{th} row of the Rencontres table, where rows and their elements are numbered beginning with 0.

Thus, by the definition of the Rencontres matrix,

$$\begin{aligned} rm_{i,j} &= rt_{i-2,j-1} \pmod 2 \text{ for } i > j \geqslant 1 \\ &= \binom{i-2}{j-1} rt_{i-j-1,0} \pmod 2 \\ &= \binom{i-2}{j-1} rm_{i-j+1,1} \pmod 2. \end{aligned}$$

Definitions 1-3 are similar to those in [5], in the context of Pascal graphs.

Definition 4: Let $BS(M)$ denote the *binary representation* of a nonnegative integer M; if q is the smallest integer such that $2^{q+1} > M$, then q will be called the *length* of $BS(M)$. The p^{th} bit of $BS(M)$ will be denoted as $BS_p(M)$, where the bits are counted from right to left and the rightmost bit is the 0^{th} bit.

Definition 5: The *B-sequence* of a positive integer N is defined as the strictly decreasing sequence $B(N) = (p_1, p_2, \ldots, p_\ell)$ of ℓ nonnegative integers such that

$$N = \sum_{i=1}^{\ell} 2^{p_i}.$$

Note that the B-sequence of any positive integer N gives the positions of 1's in the binary representation of N in decreasing order. Also, the B-sequence of zero is defined to be a null sequence. This definition is the same as in [6].

III. CONNECTIVITY PROPERTIES OF THE RENCONTRES GRAPHS

Lemma 1: Graph $RG(n)$ is a subgraph of $RG(n+1)$ for all $n \geqslant 1$.

Proof: This property is a direct consequence of the definition of the Rencontres matrix.

Theorem 1: All graphs $RG(i)$, $1 \leqslant i \leqslant 7$, are planar; all Rencontres graphs of higher order are nonplanar.

Proof: Figure 1 clearly shows that all graphs $RG(i)$ for $1 \leqslant i \leqslant 7$ are planar. It is easy to see that Kuratowski's second graph $K_{3,3}$ is a subgraph of $RG(8)$. Thus, by Lemma 1, all graphs of order 8 and higher are nonplanar.

Theorem 2: (a) Vertex v_i is adjacent to v_{i+1} in the Rencontres graph for every $i \geqslant 1$.
(b) Vertex v_1 is adjacent only to all even-numbered vertices in the Rencontres graph.
(c) Vertex v_2 is adjacent only to all odd-numbered vertices in the Rencontres graph.

Proof: (a) By the definition of the Rencontres matrix,

$$rm_{i,j} = rt_{i-2,j-1} \pmod 2, \; i > j \geqslant 1.$$

For all $i \geqslant 1$, $rm_{i+1,i} = rt_{i-1,i-1} \pmod 2 = 1$. Thus, vertex v_i is adjacent to v_{i+1} for all $i \geqslant 1$.

(b) Since $rm_{2,1} = rt_{0,0} \pmod 2 = 1$, so vertex v_1 is adjacent to v_2.

For $i \geqslant 3$, $rm_{i,1} = rt_{i-2,0} \pmod 2$

$$= (i - 2)rt_{i-3,0} + (-1)^{i-2} \pmod 2$$

$$= (i - 2)rm_{i-1,1} \pmod 2 + (-1)^{i-2} \pmod 2 \pmod 2.$$

Now, if i is even,

$$(i - 2) \pmod 2 = 0 \quad \text{and} \quad (-1)^{i-2} = 1,$$

so that $rm_{i,1} = 1$ for all even $i \geqslant 2$. On the other hand, if i is odd,

$$(i - 2) \pmod 2 = 1 \quad \text{and} \quad (-1)^{i-2} = -1;$$

also, since $i - 1$ is even, $rm_{i-1,1} = 1$. Hence, $rm_{i,1} = 0$ for all odd $i \geqslant 3$. Thus, vertex v_1 is adjacent to all even-numbered vertices and to no others in the Rencontres graph.

(c) Vertex v_2 is obviously adjacent to v_1.

For $i \geqslant 3$, $rm_{i,2} = \binom{i-2}{1} rm_{i-1,1} \pmod 2$

$$= (i - 2)rm_{i-1,1} \pmod 2.$$

Clearly, when i is even, $rm_{i,2} = 0$. But, when i is odd, $rm_{i,2} = 1$, since $rm_{i-1,1} = 1$ by Theorem 2(b). Therefore, vertex v_2 is adjacent only to all odd-numbered vertices in the Rencontres graph.

Corollary 1: Graph $RG(n)$, for all $n \geqslant 2$, is connected, and contains a Hamiltonian path $[1, 2, 3, \ldots, n]$. Moreover, for all even $n \geqslant 4$, graph $RG(n)$ contains a Hamiltonian circuit $[1, 2, \ldots, n - 1, n, 1]$.

Corollary 2:* In graph $RG(n)$, degree $(v_1) = \left\lfloor \frac{n}{2} \right\rfloor$, and degree $(v_2) = \left\lceil \frac{n}{2} \right\rceil$.

Themrem 3: $RG(n)$ is bipartite for $n \geqslant 2$.

Proof: The proof consists of showing that neither two even-numbered nor two odd-numbered vertices in a Rencontres graph are adjacent. Let both i and j be even integers, $i > j$. Then,

$$rm_{i,j} = \binom{i-2}{j-1} rm_{i-j+1,1} \pmod 2.$$

Since the integer $i - j$ is even, by Theorem 2(b) $rm_{i-j+1,1} = 0$, and therefore, $rm_{i,j} = 0$. Thus, no two even-numbered vertices in a Rencontres graph are adjacent. Similar argument shows that no two odd-numbered vertices in a Rencontres graph are adjacent.

Corollary 3: Since $RG(4)$ is a 4-cycle, the girth of the Rencontres graph $RG(n)$ is 4 for all $n > 3$.

Theorem 4: Vertex v_i is adjacent to v_{i+3} in the Rencontres graph iff i is 1 or 2 (mod 4).

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*$\lceil a \rceil$ *is the least integer greater than or equal to* a. $\lfloor a \rfloor$ *is the greatest integer less than or equal to* a.

Proof: $rm_{i+3,i} = \binom{i+1}{i-1} rm_{4,1} \pmod 2$

$$= \binom{i+1}{i-1} \pmod 2, \text{ by Theorem 2(b)}$$

$$= \frac{i(i+1)}{2} \pmod 2$$

$$= 1, \text{ iff } i \text{ is 1 or 2} \pmod 4.$$

The following theorem gives a necessary and sufficient condition for any two vertices to be adjacent in a Rencontres graph.

Theorem 5: Vertex v_i is adjacent to v_j, where $i > j$ and one is odd and the other even, iff there does not exist an integer p, $0 \leqslant p \leqslant k$, such that

$$BS_p(i-2) = 0 \quad \text{and} \quad BS_p(j-1) = 1,$$

where k is the length of $BS(j-1)$.

Proof: We have

$$rm_{i,j} = \binom{i-2}{j-1} rm_{i-j+1,1} \pmod 2.$$

If one of i and j is odd and the other even, by Theorem 2(b) $rm_{i-j+1,1} = 1$. Thus, we have to determine the condition under which

$$\binom{i-2}{j-1} \pmod 2 = 1$$

so that vertex v_i is adjacent to v_j. Let

$$BS(i-2) = m_q m_{q-1} \ldots m_1 m_0 \quad \text{and} \quad BS(j-1) = n_k n_{k-1} \ldots n_1 n_0,$$

where $q \geqslant k$. Following [2], we can write:

$$\binom{i-2}{j-1} \pmod 2 = \binom{m_k}{n_k}\binom{m_{k-1}}{n_{k-1}} \cdots \binom{m_1}{n_1}\binom{m_0}{n_0} \pmod 2$$

$$= \begin{cases} 1 & \text{iff } m_i \geqslant n_i,\ 0 \leqslant i \leqslant k \\ 0 & \text{iff } \exists p,\ 0 \leqslant p \leqslant k \ni m_p < n_p, \\ & \text{i.e., } m_p = 0 \text{ and } n_p = 1. \end{cases}$$

Thus, $rm_{i,j} = 1$ iff there does not exist an integer p, $0 \leqslant p \leqslant k$, such that

$$BS_p(i-2) = 0 \quad \text{and} \quad BS_p(j-1) = 1,$$

where k is the length of $BS(j-1)$, and in that case vertex v_i is adjacent to v_j.

Theorem 6: If $i = 2^k + 1$, where $k \geqslant 1$, then vertex v_i is adjacent to all even-numbered vertices v_j, $2 \leqslant j < 2i$, $j \neq i$.

Proof: Let $i = 2^k + 1$, $k \geqslant 1$. Since i is odd, j must be even, if vertex v_i is adjacent to v_j.

Case 1. $2 \leqslant j < i$

$$rm_{i,j} = \binom{2^k - 1}{j-1} rm_{i-j+1,1} \pmod 2 = 1, \text{ by Theorems 2(b) and 5.}$$

Case 2. $i < j < 2i$

$$rm_{i,j} = rm_{j,i} = \binom{j-2}{2^k} rm_{i-j+1,1} \pmod 2 = 1, \text{ by Theorems 2(b) and 5.}$$

Since, for all even j, $2 \leqslant j < 2i$ and $j \neq i$, $rm_{i,j} = 1$, vertex v_i is adjacent to all such v_j.

Corollary 4: If $i = 2^k + 1$, $k \geqslant 1$, then degree $(v_i) = 2^{k-1}$ in graph $RG(i)$, and degree $(v_i) = 2^k$ in graph $RG(2^{k+1})$.

Theorem 7: If $i = 2^k$, where k is a positive integer, then vertex v_i is adjacent to all odd-numbered vertices in the Rencontres graph.

Proof: Let $i = 2^k$, where $k \geqslant 1$. Since i is even, j must be odd for adjacency. We have

$$rm_{i,j} = \binom{2^k - 2}{j - 1} rm_{i-j+1,1} \pmod 2 = 1, \text{ by Theorems 2(b) and 5.}$$

Since, for all odd j, $1 \leqslant j < i$, $rm_{i,j} = 1$, vertex v_i is adjacent to all such v_j.

Corollary 5: If $i = 2^k$, $k \geqslant 1$, then

(a) degree $(v_i) = 2^{k-1}$ in graph $RG(i)$,

(b) degree $(v_i) = 2^{k-1} + 1$ in graph $RG(2i)$.

Proof: (a) Follows directly from Theorem 7.

(b) Theorem 7 considers the adjacency of vertex v_i with v_j, $1 \leqslant j < i$. Here we also need to consider odd j such that $i < j \leqslant 2i$. In this case,

$$rm_{i,j} = \binom{j-1}{2^k - 1} rm_{j-i+1,1} \pmod 2 = 0 \text{ except when } j = 2^k + 1,$$

by Theorem 5. That is, for $i < j \leqslant 2i$, vertex v_i is adjacent to v_{i+1} only. Hence, degree $(v_i) = 2^{k-1} + 1$ in graph $RG(2i)$.

Theorem 8: If $i = 2^k + 2$, $k \geqslant 1$, then vertex v_i is adjacent to v_1, v_{i-1}, and all odd-numbered vertices v_j, $i < j < 2^{k+1}$.

Proof: Let $i = 2^k + 2$, where k is a positive integer. That v_i is adjacent to v_1 and v_{i-1} is evident by Theorems 2(a) and 2(b).

Case 1. $1 < j < i - 1$, and j is odd.

$$rm_{i,j} = \binom{2^k}{j-1} rm_{i-j+1,1} \pmod 2 = 0, \text{ by Theorem 5.}$$ Thus, v_i is not adjacent to any odd-numbered vertex v_j, $1 < j < i - 1$.

Case 2. $i < j < 2^{k+1}$, and j is odd.

$$rm_{i,j} = \binom{j-2}{2^k + 1} rm_{i-j+1,1} \pmod 2 = 1, \text{ by Theorem 5.}$$

Hence the theorem.

Corollary 6: If $i = 2^k + 2$, $k \geqslant 1$, then

(a) degree $(v_i) = 2$ in graph $RG(i)$,

(b) degree $(v_i) = 2^{k-1} + 1$ in graph $RG(2^{k+1})$.

Proof: (a) Follows from Theorems 2(a), 2(b), and Case 1 of Theorem 8.

(b) By Theorem 8, in graph $RG(2^{k+1})$, vertex v_i is adjacent to v_1, v_{i-1}, and $2^{k-1} - 1$ even-numbered vertices v_j, $i < j < 2^{k+1}$. Therefore, degree $(v_i) = 2^{k-1} + 1$ in $RG(2^{k+1})$.

The following theorem identifies the subset of Rencontres graphs which contain complete bipartite graphs as subgraphs.

Theorem 9: Complete bipartite graph $K_{t,t}$ is a subgraph of $RG(2^t)$ for all $t \geqslant 1$.

Proof: By Theorem 3, $RG(2^t)$ is a bipartite graph with the following partitioning of its vertex set,

$$V_1 = \{v_{2m+1} | 0 \leqslant m < 2^{t-1}\} \quad \text{and} \quad V_2 = \{v_{2m} | 1 \leqslant m \leqslant 2^{t-1}\}.$$

Now, choose $V'_{t1} \subset V_1$, and $V'_{t2} \subset V_2$ such that

$$V'_{t1} = \{v_1\} \cup \{v_{2^i+1} | 0 < i < t\} \quad \text{and} \quad V'_{t2} = \{v_{2^i} | 1 \leqslant i \leqslant t\}.$$

We shall prove by induction that $K_{t,t}$ is a subgraph of $RG(2^t)$, and consists of sets V'_{t1} and V'_{t2}.

Basis. Graph $K_{1,1}$ is identical to $RG(2)$. Thus, the theorem is true for $t = 1$.

Induction Hypothesis. Let the theorem be true for $t = j \geqslant 1$, i.e., $k_{j,j}$ is a subgraph of $RG(2^j)$, and the vertex sets V'_{j1} and V'_{j2} are well defined.

Induction Step. To prove it to be true for $t = j + 1$, define

$$V'_{j+1,1} = V'_{j1} \cup \{v_{2^j+1}\} \quad \text{and} \quad V'_{j+1,2} = V'_{j2} \cup \{v_{2^{j+1}}\}.$$

Then, by Theorem 6, the vertex v_{2^j+1} is adjacent to all even-numbered vertices and, by Theorem 7, the vertex $v_{2^{j+1}}$ is adjacent to all odd-numbered vertices in $K_{j,j}$. Hence, we obtain the graph $K_{j+1,j+1}$, which is a subgraph of $RG(2^{j+1})$.

The following connectivity properties are useful in the design of reliable communication and computer networks. From Theorems 2(b), 2(c), 6, and 7, we conclude that vertices v_1 and $v_{2^{\lceil \log n \rceil - 1}+1}$ always serve as two central vertices adjacent to all even-numbered vertices in graph $RG(n)$; and v_2 is always the central vertex adjacent to all odd-numbered vertices in $RG(n)$. Moreover, when $n = 2^k$, $k \geqslant 1$, vertices v_2 and v_n are centrally adjacent to all odd-numbered vertices in $RG(n)$.

Theorem 10: There are at least two edge-disjoint paths of length $\leqslant 3$ between any two distinct vertices in graph $RG(n)$, $n \geqslant 4$.

Proof: Let v_i and v_j be two vertices of graph $RG(n)$, $n \geqslant 4$, $i \neq j$.

Case 1. $i = 1$ and $j = 2$

Two edge-disjoint paths are $[v_1, v_2]$ and $[v_1, v_4, v_3, v_2]$.

Case 2. $i = 1$ and $j > 2$

Two edge-disjoint paths are $[v_1, v_j]$ and $[v_1, v_{j+2}, v_{j+1}, v_j]$ for j even; and $[v_1, v_2, v_j]$ and $[v_1, v_{j-1}, v_j]$ for j odd.

Case 3. $i > 2$ and $j > 2$

If there is an edge between v_i and v_j, then it constitutes one path. Even if there is no such edge, we have the following two edge-disjoint paths in different subcases.

(i) i even and j odd

$[v_i, v_{i-1}, v_2, v_j]$ and $[v_i, v_1, v_{j-1}, v_j]$

(ii) i odd and j even

$[v_i, v_{i-1}, v_1, v_j]$ and $[v_i, v_2, v_{j-1}, v_j]$

(iii) i even and j even

$[v_i, v_1, v_j]$ and $[v_i, v_{2^{\lceil \log n \rceil - 1}+1}, v_j]$

(iv) i odd and j odd

$[v_i, v_2, v_j]$ and $[v_i, v_{2^{\lceil \log n \rceil}}, v_j]$ if i and $j \leqslant 2^{\lceil \log n \rceil} + 1$

or

$[v_i, v_2, v_j]$ and $[v_i, v_{2^{\lceil \log n \rceil}+2}, v_j]$ if i and $j \geqslant 2^{\lceil \log n \rceil} + 3$

Theorem 10 implies that the edge-connectivity $\geqslant 2$ and that the diameter is 3 for all $RG(n)$, $n \geqslant 4$.

IV. NUMBER OF EDGES IN RENCONTRES GRAPHS

Since the cost of a communication network is proportional to the number of edges in the graph (these edges represent the full duplex communication lines among processors), an estimation of the number of edges in graph $RG(n)$ is important. In the following, we derive an expression for the number of edges in $RG(n)$ in terms of n, the number of vertices in the graph. Before doing this, we need some lemmas.

Lemma 2: If $n = 2^k + i$, $k \geqslant 1$ and $1 < i \leqslant 2^k$, then $d(n) = 2 \cdot d(i)$, where $d(n)$ is the degree of vertex v_n in $RG(n)$ and $d(i)$ is the degree of vertex v_i in $RG(i)$.

Proof: Let i and j have different parity. For $1 \leqslant j < i$, we have

$$rm_{i,j} = \binom{i-2}{j-1} rm_{i-j+1,1} \pmod 2$$

$$= \binom{i-2}{j-1} \pmod 2, \text{ by Theorem 2(b).}$$

Let q be the length of $BS(j-1)$. Then, by Theorem 5,

$$d(i) = \sum_{1 \leqslant j < i} \left[\binom{i-2}{j-1} \pmod 2\right]$$

$$= \text{the number of } j\text{'s, } 1 \leqslant j < i, \text{ for which}$$

$$BS_p(i-2) \geqslant BS_p(j-1), \text{ for } 0 \leqslant p \leqslant q.$$

Now, let $n = 2^k + i$, $k \geqslant 1$ and $1 < i \leqslant 2^k$. Let $2^k + i$ and r have different parity. Then, for $1 \leqslant r < n$, we have

$$rm_{n,r} = \binom{n-2}{r-1} rm_{n-r+1,1} \pmod 2 = \binom{2^k + i - 2}{r - 1} \pmod 2.$$

Clearly,

$$d(n) = \sum_{1 \leqslant r < n} \left[\binom{2^k + i - 2}{r - 1} \pmod 2\right]$$

$= 2$ times the number of j's, $1 \leqslant j < i$, for which $BS_p(i - 2) \geqslant BS_p(j - 1)$ for each p, $0 \leqslant p \leqslant q$.

This is because $BS_k(2^k + i - 2) = 1$ and $BS_k(r - 1)$ can be 0 or 1, while

$$BS_k(i - 2) = BS_k(j - 1) = 0 \text{ (always)}.$$

Thus, $d(n) = 2 \cdot d(i)$ for all i, $1 < i \leqslant 2^k$ and $k \geqslant 1$.

Corollary 7: If $n = 2^k + 1 + i$, for $k \geqslant 1$ and $1 \leqslant i \leqslant 2^k$, then the degree $d(n)$ of vertex v_n in $RG(n)$ is given by

$$d(n) = 2 \cdot d(i + 1),$$

where $d(i + 1)$ is the degree of vertex v_{i+1} in $RG(i + 1)$.

Proof: This corollary is identical to Lemma 2 for all i, $1 \leqslant i < 2^k$. Hence, to prove this corollary, we need to consider another case where $i = 2^k$. In that case, $n = 2^{k+1} + 1$, and by Corollary 4, $d(n) = 2^k$ and $d(i + 1) = 2^{k-1}$. Thus, $d(n) = 2 \cdot d(i + 1)$ for all i such that $1 \leqslant i \leqslant 2^k$ and $k \geqslant 1$.

Lemma 3: Define $e(n)$ to be the number of edges in the bipartite graph $RG(n)$. Then

$$e(2^k) = \begin{cases} 3 \cdot e(2^{k-1}) + 2^{k-2}, & k > 1 \\ 1, & k = 1 \end{cases} \tag{1}$$

Proof: When $k = 1$, $e(2) = 1$ is obviously true. Let $n = 2^k$, $k > 1$. Then,

$e(2^k) = e(2^{k-1})$ + the number of edges added because of the addition of extra 2^{k-1} vertices, e.g., $v_{(n/2)+1}, v_{(n/2)+2}, \ldots, v_n$

$= e(2^{k-1})$ + the number of edges added because of the addition of vertices $v_{(n/2)+2}, v_{(n/2)+3}, \ldots, v_n$

+ the number of edges added because of the addition of vertex $v_{2^{k-1}+1}$

$= e(2^{k-1}) + 2 \cdot e(2^{k-1}) + 2^{k-2}$, by Lemma 2 and Corollary 4.

Therefore, $e(2^k) = 3 \cdot e(2^{k-1}) + 2^{k-2}$, for $k > 1$.

Theorem 11: If $n = 2^k$, $k \geqslant 1$, then $e(n) = 2 \cdot 3^{k-1} - 2^{k-1} = \frac{2}{3} \cdot n^{\log 3} - \frac{n}{2}$.

Proof: We shall prove this theorem by solving the recurrence equation (1). Let $n = 2^k$, i.e., $k = \log n \geqslant 1$. The homogeneous solution of (1) is $e(n) = A \cdot 3^k$, where the arbitrary constant A is to be evaluated from $e(2)$. The particular solution of (1) is $e(n) = -2^{k-1}$, so the general solution for $e(n)$ is given by

$$e(n) = A \cdot 3^k - 2^{k-1}.$$

Since $e(2) = 1$ yields $A = 2/3$, we have

$$e(n) = 2 \cdot 3^{k-1} - 2^{k-1} = \frac{2}{3} \cdot n^{\log 3} - \frac{n}{2}.$$

Corollary 8: The number of edges in graph $RG(2^k - 1)$ is

$$e(2^k - 1) = e(2^k) - 2^{k-1} = 2 \cdot 3^{k-1} - 2^k, \text{ for all } k \geqslant 1.$$

Proof: Follows from Corollary 5 and Theorem 11.

Corollary 9: The number of edges in graph $RG(2^k + 1)$ is given by

$$e(2^k + 1) = e(2^k) + 2^{k-1} = 2 \cdot 3^{k-1}, \text{ for } k \geqslant 1.$$

Proof: Corollary 9 can be proved easily using Corollary 4 and Theorem 11.

Another proof can be given as follows:

$$\begin{aligned} e(2^k + 1) &= e(2^{k-1} + 1) + \text{the number of edges addes owing to the addition of extra } 2^{k-1} \text{ vertices} \\ &= e(2^{k-1} + 1) + 2 \cdot e(2^{k-1} + 1), \text{ by Corollary 7} \\ &= 3 \cdot e(2^{k-1} + 1) \\ &\ \vdots \\ &= 3^{k-1} \cdot e(3). \end{aligned}$$

Now, $e(3)$ corresponds to the number of edges in graph $RG(3)$, which is 2; thus, $e(2^k + 1) = 2 \cdot 3^{k-1}$.

The expression for $e(n)$, the number of edges in graph $RG(n)$, is different for even and odd n. We prove this in the following theorem.

Theorem 12: The number of edges in graph $RG(n)$ is given by

$$e(n) = \begin{cases} \displaystyle\sum_{i=1}^{\ell} 2^i \cdot 3^{p_i - 1}, & \text{if } n \geqslant 3 \text{ is odd} \\ \displaystyle\sum_{i=1}^{\ell-1} 2^i \cdot 3^{p_i - 1} + 2^{\ell-1}, & \text{if } n \text{ is even,} \end{cases}$$

where $B(n - 1) = (p_1, p_2, \ldots, p_\ell)$ is the B-sequence of $n - 1$.

Proof:

Case 1. Let $n \geqslant 3$ be odd. Then $n - 1 = n_1 + n_2 + \cdots + n_\ell$, where $n_i = 2^{p_i}$ with $p_i \geqslant 1$, $1 \leqslant i \leqslant \ell$. Thus,

$$\begin{aligned} e(n) &= e(n_1 + n_2 + n_3 + \cdots + n_\ell) \\ &= e(n_1 + 1) + \text{the number of edges because of the addition of vertices } v_{n_1+2}, \ldots, v_n \text{ to } RG(n_1 + 1) \\ &= 2 \cdot 3^{p_1 - 1} + 2 \cdot e(n_2 + 1 + n_3 + \cdots + n_\ell), \end{aligned}$$

by Corollaries 7 and 9.

Repeating the process, we get

$$\begin{aligned} e(n) &= 2 \cdot 3^{p_1 - 1} + 2^2 \cdot 3^{p_2 - 1} + 2^2 \cdot e(n_3 + 1 + n_4 + \cdots + n_\ell) \\ &= 2 \cdot 3^{p_1 - 1} + 2^2 \cdot 3^{p_2 - 1} + \cdots + 2^{\ell - 1} \cdot 3^{p_{\ell - 1} - 1} + 2^\ell \cdot 3^{p_\ell - 1} \\ &= \sum_{i=1}^{\ell} 2^i \cdot 3^{p_i - 1}. \end{aligned}$$

Case 2. Let n be even. Then, $n - 1 = n_1 + n_2 + \cdots + n_{\ell-1} + n_\ell$, where $n_i = 2^{p_i}$ with $p_i \geqslant 1$ for $1 \leqslant i \leqslant \ell - 1$, $p_\ell = 0$, and $n_\ell = 1$. Following the same procedure as in the proof of Case 1 of this theorem, we get

$$\begin{aligned} e(n) &= 2 \cdot 3^{p_1 - 1} + 2^2 \cdot 3^{p_2 - 1} + \cdots + 2^{\ell - 1} \cdot 3^{p_{\ell - 1} - 1} \\ &\qquad + 2^{\ell - 1} \cdot e(n_\ell + 1) \\ &= \sum_{i=1}^{\ell - 1} 2^i \cdot 3^{p_i - 1} + 2^{\ell - 1}, \text{ since } e(n_\ell + 1) = e(2) = 1. \end{aligned}$$

In Section V we shall investigate the determinants of Rencontres matrices.

V. DETERMINANTS OF RENCONTRES MATRICES

Theorem 13: Let $\det(RM(n))$ be the determinant of the Rencontres matrix $RM(n)$ of order n. Then $\det(RM(n)) = 0$ for all $n \geqslant 1$ except for $n = 2$ and $\det(RM(2)) = -1$.

Proof: $\det(RM(1))$ is obviously zero, and

$$\det(RM(2)) = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

For $n > 2$, there always exists $k \geqslant 1$ such that $k = [\log n] - 1$ and row $2^k + 1$ is identical to row 1 in matrix $RM(n)$ by Theorem 6. Therefore, $\det(RM(n)) = 0$ for all $n > 2$.

VI. CONCLUSION

We have defined Rencontres matrices, a new class of adjacency matrices constructed from the Rencontres number table modulo 2. The corresponding graphs are connected and bipartite with edge connectivity $\geqslant 2$, diameter 3, and girth 4. The number of edges $\leqslant (2/3) \cdot n^{\log 3} - (n/2)$. Since the binary representation of a vertex number provides a great deal of information on its adjacencies, the situation may be exploited (1) in economic storage of these graphs and (2) in designing a routing algorithm between a pair of communicating vertices. These are some of the desirable properties; additional properties need to be studied to determine how well these graphs are suited for computer interconnection networks.

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