VERIFYING AND GENERALIZING ARNDT'S COMPOSITIONS

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Abstract. In 2013, Joerg Arndt recorded that the Fibonacci numbers count integer compositions where the first part is greater than the second, the third part is greater than the fourth, etc. We provide two combinatorial proofs that verify his observation. Also, we generalize the descent condition and establish families of recurrence relations. Compositions with parts restricted to 1 and 2 play an important role.

1. Introduction

Joerg Arndt observed the following occurrence of the Fibonacci numbers counting a subset of integer compositions. This appears as a comment in the On-Line Encyclopedia of Integer Sequences [2, A000045].

An integer composition of a positive integer n is an ordered collection of parts $(c_1, c_2, \ldots, c_t)$ such that $\sum c_i = n$. When listing compositions with single-digit parts, we often use the condensed representation $c_1c_2\cdots c_t$. Let $C(n)$ be the set of all compositions of n. We define Arndt's compositions in terms of pairwise descending parts.

Definition 1.1. *Let $A(n) \subset C(n)$ be the compositions such that $c_{2i-1} > c_{2i}$ for each positive integer i. If the number of parts is odd, then the final inequality is vacuously true.*

See Table 1 for examples. Arndt recorded that $a(n) = |A(n)| = f_n$, the nth Fibonacci number defined by $f_0 = 0$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$, but did not provide a proof (and, per personal communication in April 2022, does not recall how he made the connection).

We establish Arndt's observation in two ways and analyze the generalization to compositions where the restriction is modified to $c_{2i-1} > c_{2i} + k$ for any nonnegative integer k. We will use the following restricted type of compositions which are also counted by the Fibonacci numbers.

Definition 1.2. *Let $C_{12}(n) \subset C(n)$ be the compositions with parts restricted to the set $\{1, 2\}$.*

It is well known that $|C_{12}(n)| = f_{n+1}$; to see that $C_{12}(n-1) \cup C_{12}(n-2) \cong C_{12}(n)$, add a part 1 at the end of the compositions in $C_{12}(n-1)$ and add a part 2 at the end of compositions in $C_{12}(n-2)$. The relation of these compositions to what we now call the Fibonacci numbers

n	$A(n)$	$a(n)$
1	1	1
2	2	1
3	3, 21	2
4	4, 31, 211	3
5	5, 41, 32, 311, 212	5
6	6, 51, 42, 411, 321, 312, 213, 2121	8
7	7, 61, 52, 511, 43, 421, 412, 322, 313, 3121, 214, 2131, 21211	13

Table 1. Arndt's compositions for small values of n.

was known to scholars of ancient India [3]. These compositions are also the workhorses of Benjamin and Quinn's combinatorial proofs of many Fibonacci identities [1].

2. Proofs of Arndt's observation

We show the connection of Arndt's compositions and the Fibonacci numbers in two ways. First, we establish that they satisfy the same recurrence with the same initial values. Second, we show that they are in bijection with $C_{12}(n-1)$. Both proofs are combinatorial. Both approaches are used in the next section.

Theorem 2.1. *$a(n) = f_n$ for each positive integer n.*

Proof. We proceed by induction. From Table 1, we see that $A(1)$ and $A(2)$ are both singleton sets, matching $a(1) = a(2) = 1 = f_1 = f_2$.

We establish a bijection $A(n-1) \cup A(n-2) \cong A(n)$.

Given a composition $c = (c_1, \ldots, c_t) \in A(n-1)$, let

$$c \mapsto \begin{cases} (c_1, \ldots, c_t + 1) & \text{if } t \text{ is odd,} \\ (c_1, \ldots, c_t, 1) & \text{if } t \text{ is even.} \end{cases}$$

Intuitively, we are increasing the greatest possible odd index part by 1, whether it is last part of c (odd length) or follows the last part of c (even length). Clearly the image is a composition of n. Note that the image has odd length in either case, so the descent condition is maintained and the image is in $A(n)$.

These images of $A(n-1)$ in $A(n)$ are distinct as images of odd length compositions have a final part at least 2 while images of even length compositions have final part 1.

Given a composition $c = (c_1, \ldots, c_t) \in A(n-2)$, let

$$c \mapsto \begin{cases} (c_1, \ldots, c_t + 1, 1) & \text{if } t \text{ is odd,} \\ (c_1, \ldots, c_{t-1} + 1, c_t + 1) & \text{if } t \text{ is even.} \end{cases}$$

Descriptively, this adds a 1 to the last two possible positions starting with the greatest positive odd index part. Clearly the image is a composition of n. Note that the image has even length in either case. The descent condition is maintained since, for t odd, $c_t + 1 > 1$ and, for t even, $c_{t-1} > c_t$ implies $c_{t-1} + 1 > c_t + 1$. Therefore the image is in $A(n)$.

These images of $A(n-2)$ in $A(n)$ are distinct as images of odd length compositions have a final part 1 while the images of even length compositions have final part at least 2.

Also, the images from $A(n-1)$ in $A(n)$ and from $A(n-2)$ in $A(n)$ are distinct since they have odd length versus even length, respectively.

The reverse map is clear by considering the parity of the length of the element of $A(n)$ and then whether the last part is 1 or greater. More explicitly,

$$(c_1, \ldots, c_t) \mapsto \begin{cases} (c_1, \ldots, c_t - 1) & \text{if } t \text{ is odd,} \\ (c_1, \ldots, c_{t-1} - 1, c_t - 1) & \text{if } t \text{ is even,} \end{cases}$$

where the final 0 is omitted whenever $c_t = 1$.

Since $a(n-1) = f_{n-1}$ and $a(n-2) = f_{n-2}$ by assumption, we conclude that $a(n) = f_n$. □

For example, the image of $A(6)$ in $A(7)$ described in the proof is

$$6 \mapsto 7,\ 411 \mapsto 412,\ 321 \mapsto 322,\ 312 \mapsto 313,\ 213 \mapsto 214;\ 51 \mapsto 511,\ 42 \mapsto 421,\ 2121 \mapsto 21211$$

while the image of $A(5)$ in $A(7)$ is

$$5 \mapsto 61,\ 311 \mapsto 3121,\ 212 \mapsto 2131;$$
$$41 \mapsto 52,\ 32 \mapsto 43.$$

The second conclusion of the following corollary gives an unusual combinatorial verification of the identity $f_n = f_{n-2} + 2f_{n-3} + f_{n-4}$.

Corollary 2.2. *For $n > 1$, in $A(n)$ there are f_{n-1} compositions with odd length and f_{n-2} compositions with even length. For $n > 3$, in $A(n)$ there are $2f_{n-3}$ compositions whose last part is 1 and $f_{n-2} + f_{n-4}$ compositions whose last part is at least 2.*

Proof. The first statement follows from the notes in the theorem proof that, in $A(n)$, images of $A(n-1)$ have odd length and images of $A(n-2)$ have even length.

By the descriptions of the images in the theorem proof, the compositions of $A(n)$ with last part 1 come from compositions in $A(n-1)$ with even length and compositions in $A(n-2)$ with odd length. By the first statement of the corollary, the sizes of those sets are each f_{n-3}. The complementary set of compositions in $A(n)$ whose last part is at least 2 comes from the f_{n-2} compositions in $A(n-1)$ with odd length and the f_{n-4} compositions in $A(n-2)$ with even length. □

Now we establish the connection between Arndt's compositions and compositions whose parts are restricted to 1 and 2.

Theorem 2.3. $A(n) \cong C_{12}(n-1)$.

Proof. Given $c = (c_1, \ldots, c_t) \in A(n)$, convert each pair of parts (c_{2i-1}, c_{2i}) into $(1^{c_{2i-1}-c_{2i}}, 2^{c_{2i}})$, where superscripts denote repetition, a composition in $C_{12}(c_{2i-1} + c_{2i})$. Concatenating these gives an image $c' \in C_{12}(n)$. Visually, this uses the "bar graph" representation of a composition with each part c_i represented by a column of c_i boxes. The operation reads two bars of height c_{2i-1} and c_{2i} from top to bottom by row length.

Since $c_1 > c_2$, the corresponding $c' \in C_{12}(n)$ begins with a run of 1s of length at least 1; removing the initial 1 gives the final image in $C_{12}(n-1)$. See Figure 1 for an example.

For the reverse map, given a composition in $C_{12}(n-1)$, add an additional part 1 at the beginning. The resulting composition in $C_{12}(n)$ can be broken into runs of the form $(1^a, 2^b)$ for positive integers a and b, except that the final b could be 0, i.e., the last part of the composition could be a 1. The subsequence $(1^a, 2^b)$ corresponds to the 2-part composition $(a+b, b) \in A(a+2b)$; since $a > 0$, we have $a + b > b$ as required. Concatenating the pairs (and possibly a final singleton) gives a composition in $A(n)$.

It is clear that the two maps are inverses, establishing the bijection. □

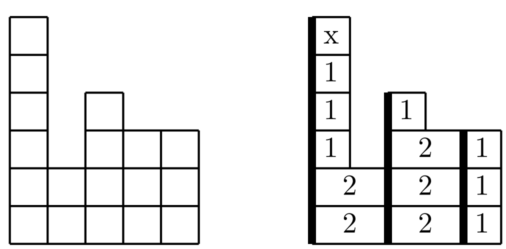

FIGURE 1. $(6, 2, 4, 3, 3) \in A(18)$ corresponds to $(1, 1, 1, 2, 2, 1, 2, 2, 2, 1, 1, 1) \in C_{12}(17)$.

For example, the element by element correspondence between $A(7)$ and $C_{12}(6)$ described in the proof is

$$7 \mapsto 1^6,\ 511 \mapsto 11121,\ 421 \mapsto 1221,\ 412 \mapsto 11211,\ 322 \mapsto 2211,$$
$$313 \mapsto 12111,\ 214 \mapsto 21^4,\ 21211 \mapsto 2121;$$
$$61 \mapsto 1^4 2,\ 52 \mapsto 1122,\ 43 \mapsto 222,\ 3121 \mapsto 1212,\ 2131 \mapsto 2112,$$

where, looking ahead to Corollary 3.6, the compositions of $A(7)$ are organized by length parity.

3. Generalizing Arndt's compositions

Arndt's compositions $A(n)$ require a descent from each c_{2i-1} to c_{2i} for each i. We generalize these compositions by requiring a greater decrease.

Definition 3.1. *Given a nonnegative integer k, let $A(n,k) \subset C(n)$ be the compositions such that $c_{2i-1} > c_{2i} + k$ for each positive integer i. If the number of parts is odd, then the final inequality is vacuously true.*

The Arndt compositions $A(n)$ are the same as $A(n,0)$. Table 2 shows some examples for small k.

In this section, we establish a recurrence relation for $a(n,k) = |A(n,k)|$ and show that these compositions are in bijection to certain subsets of $C_{12}(n-k-1)$.

Theorem 3.2. *Given a positive integer k, for all $n > k+3$,*

$$a(n,k) = a(n-1,k) + a(n-2,k) - a(n-3,k) + a(n-k-3,k).$$

Proof. We establish a bijection

$$A(n,k) \cup A(n-3,k) \cong A(n-1,k) \cup A(n-2,k) \cup A(n-k-3,k).$$

Given $c = (c_1, \ldots, c_t) \in A(n,k)$, let

$$c \mapsto \begin{cases} (c_1, \ldots, c_t - 1) & \text{if } t \text{ is odd,} \\ (c_1, \ldots, c_{t-1} - 1, c_t - 1) & \text{if } t \text{ is even,} \end{cases}$$

where the final 0 is omitted whenever $c_t = 1$. From $A(n,k)$, the odd length compositions go into $A(n-1,k)$ while the even length compositions go into $A(n-2,k)$.

Given $c = (c_1, \ldots, c_t) \in A(n-3,k)$, let

$$c \mapsto \begin{cases} (c_1, \ldots, c_t + 1) & \text{if } c_t < k \text{ and } t \text{ is odd,} \\ (c_1, \ldots, c_t, 1) & \text{if } c_t < k \text{ and } t \text{ is even,} \\ (c_1, \ldots, c_t - k) & \text{if } c_t \geq k, \end{cases}$$

where the final 0 is omitted whenever $c_t = k$. The compositions in $A(n-3,k)$ with last part less than k go into $A(n-2,k)$ while the others go into $A(n-k-3,k)$.

k	$A(7,k)$
0	7, 61, 52, 511, 43, 421, 412, 322, 313, 3121, 214, 2131, 21211
1	7, 61, 52, 511, 421, 412, 313
2	7, 61, 52, 511, 412
3	7, 61, 511

Table 2. Compositions $A(7,k)$ for small values of k.

Note that the image $c' \in A(n-2,k)$ of $c = (c_1, \ldots, c_t) \in A(n,k)$ and the image $d' \in A(n-2,k)$ of $d = (d_1, \ldots, d_s) \in A(n-3,k)$ are distinct: If $c_t > 1$, then c' and d' have different length parity. If $c_t = 1$, then $c' = (c_1, \ldots, c_{t-1} - 1)$ with last part at least $k+1$ while d' is $(d_1, \ldots, d_s + 1)$ or $(d_1, \ldots, d_s, 1)$ with last part at most k in either case.

Thus these maps give an injection from $A(n,k) \cup A(n-3,k)$ to $A(n-1,k) \cup A(n-2,k) \cup A(n-k-3,k)$.

We now describe the reverse map to complete the bijection.

Given $c = (c_1, \ldots, c_t) \in A(n-1,k)$, let

$$c \mapsto \begin{cases} (c_1, \ldots, c_t + 1) & \text{if } t \text{ is odd,} \\ (c_1, \ldots, c_t, 1) & \text{if } t \text{ is even.} \end{cases}$$

Note that all compositions from $A(n-1,k)$ go into $A(n,k)$. Further, the images of $A(n-1,k)$ in $A(n,k)$ are distinct since odd length compositions have images with final part at least 2 while even length compositions have images with final part 1.

Given $c = (c_1, \ldots, c_t) \in A(n-2,k)$, let

$$c \mapsto \begin{cases} (c_1, \ldots, c_t + 1, 1) & \text{if } t \text{ is odd and } c_t > k, \\ (c_1, \ldots, c_t - 1) & \text{if } t \text{ is odd and } c_t \le k, \\ (c_1, \ldots, c_{t-1} + 1, c_t + 1) & \text{if } t \text{ is even,} \end{cases}$$

where the final 0 is omitted whenever t is odd and $c_t = 1$.

From $A(n-2,k)$, the odd length compositions with last part greater than k and the even length compositions go into $A(n,k)$, while the odd length compositions with last part at most k go into $A(n-3,k)$.

Given $c = (c_1, \ldots, c_t) \in A(n-k-3,k)$, let

$$c \mapsto \begin{cases} (c_1, \ldots, c_t + k) & \text{if } t \text{ is odd,} \\ (c_1, \ldots, c_t, k) & \text{if } t \text{ is even.} \end{cases}$$

Note that all compositions from $A(n-k-3,k)$ go into $A(n-3,k)$. Further, the images of $A(n-k-3,k)$ in $A(n-3,k)$ are distinct since odd length compositions have images with final part at least $k+1$ while even length compositions have images with final part k.

Moreover, the images from $A(n-1,k)$ in $A(n,k)$ and from $A(n-2,k)$ in $A(n,k)$ are distinct since they have odd length versus even length, respectively. Finally, the image $c' \in A(n-k-3,k)$ of $c = (c_1, \ldots, c_t) \in A(n-2,k)$ and the image $d' \in A(n-k-3,k)$ of $d = (d_1, \ldots, d_s) \in A(n-k-3,k)$ are distinct: If $c_t > 1$, then $c' = (c_1, \ldots, c_t - 1)$ with last part at most $k-1$ while d' is $(d_1, \ldots, d_s + k)$ or $(d_1, \ldots, d_s, k)$ with last part at least k in either case. If $c_t = 1$, then $c' = (c_1, \ldots, c_{t-1})$ with even length while d' has odd length.

Therefore, the map from $A(n-1,k) \cup A(n-2,k) \cup A(n-k-3,k)$ to $A(n,k) \cup A(n-3,k)$ is an injection and the desired bijection is established. $\square$

For example, the correspondence between $A(9,2) \cup A(6,2)$ and $A(8,2) \cup A(7,2) \cup A(4,2)$ is

$$9 \mapsto 8,\ 711 \mapsto 71,\ 621 \mapsto 62,\ 612 \mapsto 611,\ 522 \mapsto 521,\ 513 \mapsto 512,\ 414 \mapsto 413;$$
$$81 \mapsto 7,\ 72 \mapsto 61,\ 63 \mapsto 52;\ 51 \mapsto 511;\ 411 \mapsto 412;\ 6 \mapsto 4.$$

Similar to Corollary 2.2, the map of Theorem 3.2 gives the following final result.

Corollary 3.3. *The number of odd length compositions in $A(n,k)$ is $a(n-1,k)$. The number of even length compositions in $A(n,k)$ is then $a(n,k) - a(n-1,k)$.*

Finally, we show that the compositions $A(n,k)$ are in bijection to particular subsets of $C_{12}(n-k-1)$.

Definition 3.4. *Given a nonnegative integer k, let $C_{12}^k(n)$ denote that subset of $C_{12}(n)$ where each internal run of 1s has length at least $k+1$. In other words, given $c=(c_1,\ldots,c_t)\in C_{12}^k(n)$, a run of 1s starting with c_1 or ending with c_t may be shorter, but any run of 1s between two 2s must have length greater than k.*

We can write $C_{12}(n)=C_{12}^0(n)$. For positive k, the set $C_{12}^k(n)$ can be described in terms of forbidden subwords: $C_{12}^1(n)$ consists of the compositions in $C_{12}(n)$ that avoid $(2,1,2)$ and $C_{12}^2(n)$ are the compositions of $C_{12}(n)$ that avoid both $(2,1,2)$ and $(2,1,1,2)$, etc. The sequence $|C_{12}^1(n)|$ is [2, A130137] with an equivalent forbidden word interpretation; $|C_{12}^2(n)|$ is the positive version of [2, A107332].

Theorem 3.5. $A(n,k)\cong C_{12}^k(n-k-1)$.

Proof. The same map in the proof of Theorem 2.3 works here. Given $c=(c_1,\ldots,c_t)\in A(n,k)$, since $c_{2i-1}>c_{2i}+k$ for each i, the same conversion into $C_{12}(n)$ has each run of 1s, except possibly the last, with length at least $k+1$. Removing the initial $k+1$ parts 1 leaves a composition in $C_{12}^k(n-k-1)$. In the reverse map, given a composition in $C_{12}^k(n-k-1)$, add $k+1$ parts 1 at the beginning and proceed as before. □

Examples are essentially subsets of the $A(7)$ and $C_{12}(6)$ example above. For instance, for $A(7,1)$ and $C_{12}^1(5)$ we have

$$7\mapsto 1^5,\ 511\mapsto 1121,\ 421\mapsto 221,\ 412\mapsto 1211, 313\mapsto 2111;\ 61\mapsto 1112,\ 52\mapsto 122,$$

while for $A(7,2)$ and $C_{12}^2(4)$ the element by element correspondence is

$$7\mapsto 1^4,\ 511\mapsto 121,\ 412\mapsto 211; 61\mapsto 112,\ 52\mapsto 22.$$

Related to the first statement of Corollary 2.2, we have the following result.

Corollary 3.6. *For nonnegative k, the number of compositions in $A(n,k)$ with odd length equals the number of compositions in $C_{12}^k(n-k-1)$ whose last part is 1. Therefore the number of compositions in $A(n,k)$ with even length equals the number of compositions in $C_{12}^k(n-k-1)$ whose last part is 2.*

Proof. This follows directly from the correspondence used in Theorems 2.3 and 3.5. □

In future work, we plan to expand Definition 3.1 to negative values of k and explore connections to Carlitz compositions.

References

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