THE PASCAL RHOMBUS AND RIORDAN ARRAYS

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Abstract. In the present article, we first obtain Riordan array expressions for the right half of the Pascal rhombus and the left-bounded rhombus. Then, a combinatorial interpretation based on the 2-generalized Motzkin paths is given for these arrays. Moreover, using the k-generalized Motzkin paths, we introduce the concept of k-generalized Pascal rhombus and left-bounded rhombus. Finally, explicit formulas for the generic elements and row sums of the k-generalized Pascal rhombus and left-bounded rhombus are obtained in terms of k-Bonacci numbers.

1. Introduction

The Pascal rhombus was introduced in 1997 by Klostermeyer, et al. [8] as a generalization of the Pascal triangle. It is an infinite array $\mathcal{R} = (r_{i,j})$ where i is a non-negative integer and j is an integer, i.e., $i \in \mathbb{N}$ and $j \in \mathbb{Z}$, $r_{i,j}$ defined by

$$\begin{cases} r_{i,j} &= r_{i-1,j-1} + r_{i-1,j} + r_{i-1,j+1} + r_{i-2,j},\ i \geq 2,\ j \in \mathbb{Z}, \\ r_{0,0} &= r_{1,-1} = r_{1,0} = r_{1,1} = 1, r_{0,j} = 0\,(j \neq 0),\ r_{1,j} = 0\,(j \neq -1, 0, 1). \end{cases} \tag{1.1}$$

They also introduced the left-bounded rhombus $S = (s_{i,j})$ where $i, j \in \mathbb{N}$, $s_{i,j}$ defined by the analogue rules

$$\begin{cases} s_{i,j} &= s_{i-1,j-1} + s_{i-1,j} + s_{i-1,j+1} + s_{i-2,j},\ i \geq 2,\ 0 \leq j \leq i, \\ s_{0,0} &= s_{1,0} = s_{1,1} = 1, s_{i,-1} = 0\,(i \geq 0),\ r_{i,j} = 0\,(i < j). \end{cases} \tag{1.2}$$

The first few rows of the Pascal rhombus are given in the left of Figure 1, and the left-bounded rhombus in the right.

						1						
					1	1	1					
				1	2	4	2	1				
			1	3	8	9	8	3	1			
		1	4	13	22	29	22	13	4	1		
	1	5	19	42	72	82	72	42	19	5	1	
1	6	26	70	146	218	255	218	146	70	26	6	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

1						
1	1					
3	2	1				
6	7	3	1			
16	18	12	4	1		
40	53	37	18	5	1	
109	148	120	64	25	6	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Figure 1. Pascal Rhombus and Left-bounded Rhombus

Klostermeyer, et al. [8] studied several properties of the Pascal rhombus and the related left-bounded rhombus. They conjectured that the limiting ratio of the number of ones to the number of zeros in Pascal rhombus, taken modulo 2, approaches zero. This conjecture was settled affirmatively by Goldwasser, et al. [6] and also generalized by Mosche [11]. Stockmeyer [18] proved four conjectures about the Pascal rhombus modulo 2 given in [8].

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Recently, Ramírez found a closed expression for the entries of the Pascal rhombus in [13]. He also show a relation between the entries of the Pascal rhombus and a family of generalized grand Motzkin paths.

The aim of this paper is to establish the connection between the Pascal rhombus and the Riordan array. In Section 2, we first recall the concept of Riordan array. Then, we give Riordan array expressions for the right half of the Pascal rhombus and the left-bounded rhombus. In Section 3, a combinatorial description is carried out to give an interpretation of the Pascal rhombus and the left-bounded rhombus in terms of the 2-generalized Motzkin paths. Moreover, using the k-generalized Motzkin paths, we introduce the concept of k-generalized Pascal rhombus and left-bounded rhombus. Finally, explicit formula for the generic elements and row sums of the k-generalized Pascal rhombus and left-bounded rhombus are obtained.

2. A Riordan Array Description of the Pascal Rhombus

We will encounter Riordan arrays in this paper. So, we briefly recall the notion of Riordan arrays [16, 4, 7, 9]. An infinite lower triangular matrix $G = (g_{n,k})_{n,k\in\mathbb{N}}$ is called a Riordan array if its column k has generating function $d(t)h(t)^k$, where $d(t) = \sum_{n=0}^{\infty} d_n t^n$ and $h(t) = \sum_{n=1}^{\infty} h_n t^n$ are formal power series with $d_0 \neq 0$ and $h_1 \neq 0$. The Riordan array corresponding to the pair $d(t)$ and $h(t)$ is denoted by $(d(t), h(t))$, and its generic entry is $g_{n,k} = [t^n]d(t)h(t)^k$, where $[t^n]$ denotes the coefficient operator.

The set of all Riordan arrays forms a group under ordinary row-by-column product with the multiplication identity $(1, t)$. The product of two Riordan arrays is given by

$$(d(t), h(t))(g(t), f(t)) = (d(t)g(h(t)), f(h(t))), \tag{2.1}$$

and the inverse of $(d(t), h(t))$ is the Riordan array

$$(d(t), h(t))^{-1} = (1/d(\bar{h}(t)), \bar{h}(t)), \tag{2.2}$$

where $\bar{h}(t)$ is compositional inverse of $h(t)$, i.e., $h(\bar{h}(t)) = \bar{h}(h(t)) = t$.

If $(b_n)_{b\in\mathbb{N}}$ is any sequence having $b(t) = \sum_{n=0}^{\infty} b_n t^n$ as its generating function, then for every Riordan array $(d(t), h(t)) = (g_{n,k})_{n,k\in\mathbb{N}}$

$$\sum_{k=0}^{n} g_{n,k} b_k = [t^n]d(t)b(h(t)). \tag{2.3}$$

This is called the fundamental theorem of Riordan arrays, and it can be rewritten as

$$(d(t), h(t))b(t) = d(t)b(h(t)). \tag{2.4}$$

A characterization of Riordan arrays was established by Merlini, et al. [10] as follows.

Lemma 2.1. *A lower triangular array $(g_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array if and only if there exists another array $(\alpha_{i,j})_{i,j\in\mathbb{N}}$, with $\alpha_{0,0} \neq 0$, and s sequences $\{\rho_j^{[i]}\}_{j\in\mathbb{N}}$, $i = 1, 2, \ldots, s$, such that*

$$g_{n+1,k+1} = \sum_{i\geq 0}\sum_{j\geq 0} \alpha_{i,j} g_{n-i,k+j} + \sum_{i=1}^{s}\sum_{j\geq 0} \rho_j^{[i]} g_{n+i,k+i+j+1}. \tag{2.5}$$

The array $(\alpha_{i,j})_{i,j\in\mathbb{N}}$ in this lemma is called the A-matrix of the Riordan array $(g_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$. If $\Phi^{[i]}(t)$ denotes the generating functions of ith row of the A-matrix and $\Psi^{[i]}(t)$

is the generating function for the sequence $\{\rho_j^{[i]}\}_{j\in\mathbb{N}}$, then $h(t)$ is determined by [10]

$$h(t) = \sum_{i\geq 0} t^{i+1}\Phi^{[i]}(h(t)) + \sum_{i=1}^{s} t^{1-i}h(t)^{i+1}\Psi^{[i]}(h(t)). \tag{2.6}$$

If column 0 of the Riordan array $(g_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ is defined by

$$g_{n+1,0} = \sum_{i\geq 0}\sum_{j\geq 0} \beta_{i,j} g_{n-i,j} + \sum_{i=1}^{s}\sum_{j\geq 0} \eta_j^{[i]} g_{n+i,i+j+1},\ n \geq 0, \tag{2.7}$$

then the function $d(t)$ is given by the following formula:

$$d(t) = \frac{g_{0,0}}{1 - \sum_{i\geq 0} t^{i+1} R^{[i]}(h(t)) - t\sum_{i=1}^{s} t^{1-i}h(t)^{i} S^{[i]}(h(t))}, \tag{2.8}$$

where $R^{[i]}(t) = \sum_{j\geq 0}\beta_{i,j}t^j$, $i = 0, 1, \ldots$, and $S^{[i]}(t) = \sum_{j\geq 0}\eta_j^{[i]}t^j$, $i = 0, 1, \ldots, s$.

Now, we will show that the right half of the Pascal rhombus, and the left-bounded rhombus can be represented as Riordan arrays.

Theorem 2.2. *Let $R^{(2)} = (r_{n,k})_{n,k\in\mathbb{N}}$ denote the right half of the Pascal rhombus. Then,*

$$R^{(2)} = \left(\frac{1}{\sqrt{(1-t-t^2)^2 - 4t^2}}, \frac{1-t-t^2-\sqrt{(1-t-t^2)^2-4t^2}}{2t}\right).$$

Proof. It follows from (1.1) and Lemma 2.1 that $R^{(2)} = (r_{n,k})_{n,k\in\mathbb{N}}$ is a Riordan array $(d(t), h(t))$ with the A-matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the entries $\alpha_{0,0} = \alpha_{0,1} = \alpha_{0,2} = \alpha_{1,1} = 1$, whereas the other entries are all equal to 0. Now, we can directly use (2.6) to obtain the function $h(t)$. Because $\Phi^{[0]}(t) = 1 + t + t^2$, $\Phi^{[1]}(t) = t$, $\Phi^{[i]}(t) = 0$ for $i \geq 2$, and $\Psi^{[i]}(t) = 0$ for $i \geq 1$, therefore, $h(t)$ is the solution to the equation

$$h(t) = t(1 + h(t) + h(t)^2) + t^2 h(t),$$

from which it follows $h(t) = \frac{1-t-t^2-\sqrt{(1-t-t^2)^2-4t^2}}{2t}$.

The column 0 of the Riordan array $(r_{n,k})_{n,k\in\mathbb{N}} = (d(t), h(t))$ satisfies

$$r_{i+1,0} = r_{i,0} + 2r_{i,1} + r_{i-1,0}.$$

Hence from (2.8), the function $d(t)$ is given by

$$d(t) = \frac{1}{1 - t(1 + 2h(t)) - t^2} = \frac{1}{\sqrt{(1-t-t^2)^2 - 4t^2}}.$$

□

Theorem 2.3. *The left-bounded rhombus $S^{(2)} = (s_{n,k})_{n,k\in\mathbb{N}}$ is the Riordan array*

$$S^{(2)} = \left(\frac{1-t-t^2-\sqrt{(1-t-t^2)^2-4t^2}}{2t^2}, \frac{1-t-t^2-\sqrt{(1-t-t^2)^2-4t^2}}{2t}\right).$$

Proof. The proof is similar to that of Theorem 2.2, so it was omitted. □

3. The Combinatorial Interpretation and Generalization

A Motzkin path of length n is a lattice path from $(0,0)$ to $(n,0)$ consisting of up steps $U=(1,1)$, horizontal steps $H_1=(1,0)$, and down steps $D=(1,-1)$ that never goes below the x-axis. The number of Motzkin paths of length n is the nth Motzkin number M_n, and the Motzkin numbers form the sequence A001006 in [12]. Many other examples of bijections between Motzkin paths and other combinatorial objects can be found in [2, 3, 5, 15, 17]. A grand Motzkin path of length n is a Motzkin path without the condition that it never passes below the x-axis. The number of grand Motzkin paths of length n is the nth grand Motzkin number G_n, and the sequence of grand Motzkin numbers (central trinomial coefficients) is the sequence A002426 in [12].

Let k be a positive integer. A k-generalized grand Motzkin path of length n is a lattice path from $(0,0)$ to $(n,0)$ with up steps $U=(1,1)$, down steps $D=(1,-1)$, and horizontal steps $H_i=(i,0)$, $i=1,2,\ldots,k$, and the number of these paths of length n is denoted by $r_n^{(k)}$. The set of all partial k-generalized grand Motzkin paths ending at (i,j) is denoted by $\mathcal{R}_{i,j}^{(k)}$, and $r_{i,j}^{(k)}=|\mathcal{R}_{i,j}^{(k)}|$. Then $r_{n,0}^{(k)}=r_n^{(k)}$.

A k-generalized Motzkin path of length n is a lattice path from $(0,0)$ to $(n,0)$ consisting of up steps $U=(1,1)$, down steps $D=(1,-1)$, and horizontal steps $H_i=(i,0)$, $i=1,2,\ldots,k$, and that it never goes below the x-axis. The number of k-generalized Motzkin paths of length n is denoted by $s_n^{(k)}$. A partial k-generalized Motzkin path, also called a k-generalized Motzkin path ending at (i,j), is defined as an initial segment of a k-generalized Motzkin path with terminal point (i,j). Let $\mathcal{S}_{i,j}^{(k)}$ be the set of all partial k-generalized Motzkin paths ending at (i,j), where $\mathcal{S}_{0,0}^{(k)}=\{\varepsilon\}$ and ε is the empty path. Let $s_{i,j}^{(k)}=|\mathcal{S}_{i,j}^{(k)}|$. Then $s_{n,0}^{(k)}=s_n^{(k)}$.

Ramírez [13] shows a relation between the entries of the Pascal rhombus and the 2-generalized grand Motzkin paths as follows.

Theorem 3.1. *([13]) The number of the 2-generalized grand Motzkin paths of length n and height j is equal to the entry $r_{n,j}$ in the Pascal rhombus, i.e., $r_{n,j}=|\mathcal{R}_{n,j}^{(2)}|$, where $n,j\in\mathbb{N}$.*

In Figure 2, we give an illustration of recursion of the partial 2-generalized Motzkin paths in $\mathcal{S}_{i,j}^{(2)}$. Consequently, $s_{i,j}^{(2)}=|\mathcal{S}_{i,j}^{(2)}|$ satisfies the recurrence relation and the boundary conditions of (1.2), and hence, we have the following theorem.

Theorem 3.2. *The number of the 2-generalized Motzkin paths of length n and height j is equal to the entry $s_{n,j}$ in the left-bounded rhombus, i.e., $|\mathcal{S}_{n,j}^{(2)}|=s_{n,j}$, where $n,j\in\mathbb{N}$.*

Motivated by the previous two theorems, we introduce a generalization of the Pascal rhombus as follows.

Definition 3.3. *For a fixed positive integer k, the k-generalized Pascal rhombus $\mathcal{R}^{(k)}=(r_{i,j}^{(k)})_{i\in\mathbb{N},j\in\mathbb{Z}}$ is defined by $r_{i,j}^{(k)}=|\mathcal{R}_{i,j}^{(k)}|$, and the left-bounded k-generalized rhombus $S^{(k)}=(s_{i,j}^{(k)})_{i,j\in\mathbb{N}}$ is defined by $s_{i,j}^{(k)}=|\mathcal{S}_{i,j}^{(k)}|$.*

For $k=1$, $\mathcal{R}^{(1)}$ and $S^{(1)}$ are the grand Motzkin array (trinomial coefficients) and the Motzkin triangle, as illustrated in Figure 3.

For $k=2$, $\mathcal{R}^{(2)}$ and $S^{(2)}$ are the Pascal rhombus and the left-bounded rhombus, as illustrated in Figure 1.

For $k=3$, $\mathcal{R}^{(3)}$ and $S^{(3)}$ are illustrated in Figure 4.

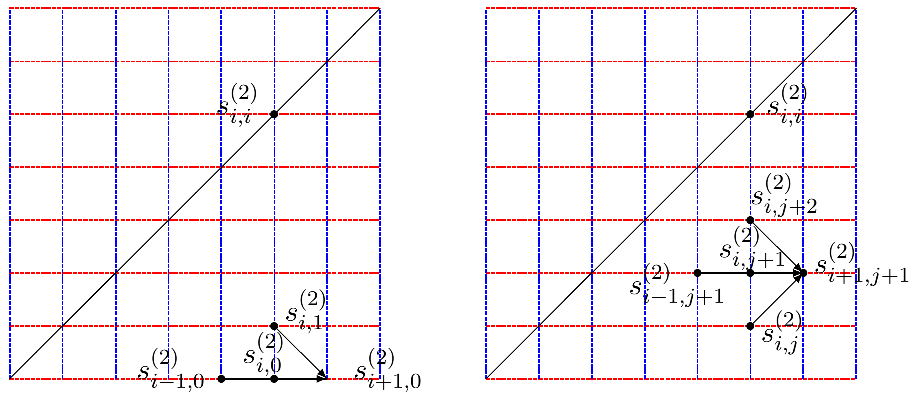

Figure 2: The recursion of the partial 2-generalized Motzkin paths

						1						
					1	1	1					
				1	2	3	2	1				
			1	3	6	7	6	3	1			
		1	4	10	16	19	16	10	4	1		
	1	5	15	30	45	51	45	30	15	5	1	
1	6	21	50	90	126	141	126	90	50	21	6	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

1						
1	1					
2	2	1				
4	5	3	1			
9	12	9	4	1		
21	30	25	14	5	1	
51	76	69	44	20	6	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Figure 3. Pascal rhombus $(r^{(1)}_{i,j})$ and left-bounded rhombus $(s^{(1)}_{i,j})$

						1						
					1	1	1					
				1	2	4	2	1				
			1	3	8	10	8	3	1			
		1	4	13	24	31	24	13	4	1		
	1	5	19	45	78	93	78	45	19	5	1	
1	6	26	74	158	248	290	248	158	74	26	6	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

1						
1	1					
3	2	1				
7	7	3	1			
18	20	12	4	1		
48	59	40	18	5	1	
132	174	132	68	25	6	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮

Figure 4. Pascal rhombus $(r^{(3)}_{i,j})$ and left-bounded rhombus $(s^{(3)}_{i,j})$

Elements in the k-generalized Pascal rhombus satisfy the following recurrence relation.

$$\begin{cases} r^{(k)}_{i,j} &= r^{(k)}_{i-1,j-1} + r^{(k)}_{i-1,j} + r^{(k)}_{i-1,j+1} + r^{(k)}_{i-2,j} + \cdots + r^{(k)}_{i-k,j},\ i \geq 2,\ j \in \mathbb{Z}, \\ r^{(k)}_{0,0} &= r^{(k)}_{1,-1} = r^{(k)}_{1,0} = r^{(k)}_{1,1} = 1, r^{(k)}_{0,j} = 0\,(j \neq 0),\ r^{(k)}_{1,j} = 0\,(j \neq -1,0,1). \end{cases} \tag{3.1}$$

Elements in the left-bounded k-generalized rhombus satisfy the same recurrence relation, but modifying by $s^{(k)}_{i,j} = 0$ for $j \leq -1$.

Theorem 3.4. *Let $R^{(k)} = (r^{(k)}_{n,j})_{n,j\in\mathbb{N}}$ be the right half of the k-generalized Pascal rhombus. Then,*

$$R^{(k)} = \left(\frac{1}{\sqrt{(1-t-\cdots-t^k)^2-4t^2}}, \frac{1-t-\cdots-t^k-\sqrt{(1-t-\cdots-t^k)^2-4t^2}}{2t} \right). \tag{3.2}$$

Proof. From recurrence relations (3.1), the array $R^{(k)} = (r^{(k)}_{n,j})_{n,j\in\mathbb{N}}$ has the A-matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where the entries $\alpha_{0,0} = \alpha_{0,1} = \alpha_{0,2} = \alpha_{1,1} = \alpha_{2,1} = \cdots = \alpha_{k,1} = 1$, whereas the other entries are all equal to 0. Hence, it follows from Lemma 2.1 that $R^{(k)} = (r^{(k)}_{n,j})_{n,j\in\mathbb{N}}$ is a Riordan array $(d(t), h(t))$. Now, we can directly use (2.6) to obtain the function $h(t)$. Because $\Phi^{[0]}(t) = 1 + t + t^2$, $\Phi^{[i]}(t) = t$ for $i = 1, \ldots, k$, and $\Phi^{[i]}(t) = 0$ for $i > k$; and $\Psi^{[i]}(t) = 0$ for $i \geq 1$, therefore, $h(t)$ is the solution to the equation

$$h(t) = t(1 + h(t) + h(t)^2) + t^2 h(t) + \cdots + t^k h(t),$$

from which it follows $h(t) = \frac{1-t-\cdots-t^k-\sqrt{(1-t-\cdots-t^k)^2-4t^2}}{2t}$.

The column 0 of the Riordan array $(r^{(k)}_{n,j})_{n,j\in\mathbb{N}} = (d(t), h(t))$ satisfies

$$r_{i+1,0} = r_{i,0} + 2r_{i,1} + r_{i-1,0} + \cdots + r_{i-k+1,0}.$$

Hence from (2.8), the function $d(t)$ is given by

$$d(t) = \frac{1}{1 - t(1 + 2h(t)) - t^2 - \cdots - t^k} = \frac{1}{\sqrt{(1 - t - \cdots - t^k)^2 - 4t^2}}.$$

This completes the proof. □

Theorem 3.5. *The k-generalized left-bounded rhombus $S^{(k)} = (s^{(k)}_{n,j})_{n,j\in\mathbb{N}}$ is the Riordan array*

$$S^{(k)} = \left(\frac{1-t-\cdots-t^k-\sqrt{(1-t-\cdots-t^k)^2-4t^2}}{2t^2}, \frac{1-t-\cdots-t^k-\sqrt{(1-t-\cdots-t^k)^2-4t^2}}{2t}\right). \tag{3.3}$$

Proof. The array $S^{(k)} = (s^{(k)}_{n,j})_{n,j\in\mathbb{N}}$ has the same A-matrix with the array $R^{(k)} = (r^{(k)}_{n,j})_{n,j\in\mathbb{N}}$. Hence, it follows from Lemma 2.1 that $S^{(k)} = (s^{(k)}_{n,j})_{n,j\in\mathbb{N}}$ is a Riordan array $(d(t), h(t))$, and $h(t) = \frac{1-t-\cdots-t^k-\sqrt{(1-t-\cdots-t^k)^2-4t^2}}{2t}$.

The column 0 of the Riordan array $S^{(k)} = (s^{(k)}_{n,j})_{n,j\in\mathbb{N}} = (d(t), h(t))$ satisfies

$$s_{i+1,0} = s_{i,0} + s_{i,1} + s_{i-1,0} + \cdots + s_{i-k+1,0}.$$

Hence from (2.8), the function $d(t)$ is given by

$$d(t) = \frac{1}{1 - t(1 + h(t)) - t^2 - \cdots - t^k} = \frac{1 - t - \cdots - t^k - \sqrt{(1 - t - \cdots - t^k)^2 - 4t^2}}{2t^2},$$

and this completes the proof. □

For example,

$$R^{(3)} = \left(\frac{1}{\sqrt{(1 - t - t^2 - t^3)^2 - 4t^2}}, \frac{1-t-t^2-t^3-\sqrt{(1-t-t^2-t^3)^2-4t^2}}{2t}\right),$$

$$S^{(3)} = \left(\frac{1 - t - t^2 - t^3 - \sqrt{(1 - t - t^2 - t^3)^2 - 4t^2}}{2t^2}, \frac{1-t-t^2-t^3-\sqrt{(1-t-t^2-t^3)^2-4t^2}}{2t}\right).$$

4. Connection with the k-Bonacci Sequence

The convolved k-Bonacci numbers $T_i^{(r)}$ are defined by [14]

$$\left(\frac{1}{1-t-\cdots-t^k}\right)^r=\sum_{i=0}^{\infty}T_i^{(r)}t^i,\ r\in\mathbb{Z}^+.$$

If $r=1$, we have the k-Bonacci sequence $(T_i)_{i\geq 0}$ with the generating function $\frac{1}{1-t-\cdots-t^k}=\sum_{i=0}^{\infty}T_it^i$. The generic entry of the Riordan array $\left(\frac{1}{1-t-\cdots-t^k},\frac{t}{1-t-\cdots-t^k}\right)$ is given by the convolved k-Bonacci number $T_{i-j}^{(j+1)}$.

Using the ordinary multinomial number $\binom{n}{j}_s$, which is defined as the jth coefficient in the development [1]

$$(1+t+t^2+\cdots+t^s)^n=\sum_{j=0}^{ns}\binom{n}{j}_s t^j,$$

we have $\left(\frac{1}{1-t-\cdots-t^k}\right)^r=\sum_{n=0}^{\infty}\binom{n+r-1}{n}t^n(1+t+\cdots+t^{k-1})^n=\sum_{n=0}^{\infty}\sum_{j=0}^{n(k-1)}\binom{n+r-1}{n}\binom{n}{j}_{k-1}t^{n+j}$. Therefore, the convolved k-Bonacci number can be written as

$$T_i^{(r)}=\sum_{j=0}^{\lfloor\frac{(k-1)i}{k}\rfloor}\binom{i-j+r-1}{i-j}\binom{i-j}{j}_{k-1}.$$

Theorem 4.1. *We have the matrix relation*

$$R^{(k)}=\left(\frac{1}{1-t-\cdots-t^k},\frac{t}{1-t-\cdots-t^k}\right)\left(\frac{1}{\sqrt{1-4t^2}},\frac{1-\sqrt{1-4t^2}}{2t}\right). \tag{4.1}$$

Moreover, $r_{i,j}^{(k)}$ *is given by the formula*

$$r_{i,j}^{(k)}=\sum_{l=j}^{i}T_{i-l}^{(l+1)}\binom{l}{\frac{l-j}{2}},\ 0\leq j\leq i. \tag{4.2}$$

Proof. By applying the product rule (2.1) and Theorem 3.5, we obtain (4.1). Since the generic element of the Riordan array $\left(\frac{1}{\sqrt{1-4t^2}},\frac{1-\sqrt{1-4t^2}}{2t}\right)$ is $b_{i,j}=\binom{i}{\frac{i-j}{2}}$, the generic entry of $R^{(k)}$ is given by $r_{i,j}^{(k)}=\sum_{l=j}^{i}T_{i,l}b_{l,j}=\sum_{l=j}^{i}T_{i-l}^{(l+1)}\binom{l}{\frac{l-j}{2}}$. □

Theorem 4.2. *The generating function for the row sums of the k-generalized Pascal rhombus* $\mathcal{R}^{(k)}=(r_{i,j}^{(k)})_{i\in\mathbb{N},j\in\mathbb{Z}}$ *is given by*

$$\sum_{n=0}^{\infty}R_n^{(k)}t^n=\frac{1}{1-3t-t^2-\cdots-t^k}. \tag{4.3}$$

Moreover, we have the formula

$$R_n^{(k)}=\sum_{j=0}^{n}T_{n-j}^{(j+1)}2^j. \tag{4.4}$$

Proof. The half of the k-generalized Pascal rhombus is the Riordan array

$$R^{(k)}=\left(\frac{1}{\sqrt{(1-t-\cdots-t^k)^2-4t^2}},\frac{1-t-\cdots-t^k-\sqrt{(1-t-\cdots-t^k)^2-4t^2}}{2t}\right).$$

By the symmetry, the row sums of $\mathcal{R}^{(k)}$ equals $R_n^{(k)} = \sum_{j=-n}^{n} r_{n,j}^{(k)} = 2\sum_{j=0}^{n} r_{n,j}^{(k)} - r_{n,0}^{(k)}$. Applying the production rule (1.1), the generating function $R(t) = \sum_{n=0}^{\infty} R_n^{(k)} t^n$ is

$$\begin{aligned} R(t) &= \left(\frac{1}{\sqrt{(1-t-\cdots-t^k)^2-4t^2}}, \frac{1-t-\cdots-t^k-\sqrt{(1-t-\cdots-t^k)^2-4t^2}}{2t}\right) \cdot \left(\frac{2}{1-t} - 1\right) \\ &= \left(\frac{1}{1-t-\cdots-t^k}, \frac{t}{1-t-\cdots-t^k}\right)\left(\frac{1}{\sqrt{1-4t^2}}, \frac{1-\sqrt{1-4t^2}}{2t}\right) \cdot \frac{1+t}{1-t} \\ &= \left(\frac{1}{1-t-\cdots-t^k}, \frac{t}{1-t-\cdots-t^k}\right) \cdot \frac{1}{1-2t} \\ &= \frac{1}{1-3t-t^2-\cdots-t^k}. \end{aligned}$$

Finally, by the last equation above, we obtain $R_n^{(k)} = \sum_{j=0}^{n} T_{n-j}^{(j+1)} 2^j$. □

Theorem 4.3. *We have the matrix relation*

$$S^{(k)} = \left(\frac{1}{1-t-\cdots-t^k}, \frac{t}{1-t-\cdots-t^k}\right)\left(\frac{1-\sqrt{1-4t^2}}{2t^2}, \frac{1-\sqrt{1-4t^2}}{2t}\right). \tag{4.5}$$

Moreover, it follows that $s_{i,j}^{(k)}$ is given by the formula

$$s_{i,j}^{(k)} = \sum_{l=j}^{i} T_{i-l}^{(l+1)} \frac{j+1}{l+1} \binom{l}{\frac{l-j}{2}}. \tag{4.6}$$

Proof. By applying the product rule (2.1) and Theorem 3.5, we obtain (4.5). Since the generic element of the Riordan array $\left(\frac{1-\sqrt{1-4t^2}}{2t^2}, \frac{1-\sqrt{1-4t^2}}{2t}\right)$ is $c_{i,j} = \frac{j+1}{i+1}\binom{i+1}{\frac{i-j}{2}}$, the generic entry of the k-generalized left-bounded Pascal rhombus is given by
$s_{i,j}^{(k)} = \sum_{l=j}^{i} T_{i,l} c_{l,j} = \sum_{l=j}^{i} T_{i-l}^{(l+1)} \frac{j+1}{l+1}\binom{l}{\frac{l-j}{2}}$. □

Theorem 4.4. *The generating function for the row sums of the k-generalized left-bounded rhombus $S^{(k)} = (s_{i,j}^{(k)})_{i,j\in\mathbb{N}}$ is given by*

$$\sum_{n=0}^{\infty} S_n^{(k)} t^n = \frac{1}{\sqrt{(1-t-\cdots-t^k)^2-4t^2}}\left(1 + \frac{1-t-\cdots-t^k-\sqrt{(1-t-\cdots-t^k)^2-4t^2}}{2t}\right). \tag{4.7}$$

Moreover, we have the formula

$$S_n^{(k)} = \sum_{j=0}^{n} T_{n-j}^{(j+1)} \binom{j}{\lfloor \frac{j}{2} \rfloor}. \tag{4.8}$$

Proof. By applying

$$\left(\frac{1-\sqrt{1-4t^2}}{2t^2}, \frac{1-\sqrt{1-4t^2}}{2t}\right)\frac{1}{1-t} = \frac{1}{\sqrt{1-4t^2}}\left(1 + \frac{1-\sqrt{1-4t^2}}{2t}\right),$$

we have

$$\begin{aligned} S(t) &= \left(\frac{1-t-\cdots-t^k-\sqrt{(1-t-\cdots-t^k)^2-4t^2}}{2t^2}, \frac{1-t-\cdots-t^k-\sqrt{(1-t-\cdots-t^k)^2-4t^2}}{2t}\right)\cdot\frac{1}{1-t} \\ &= \left(\frac{1}{1-t-\cdots-t^k}, \frac{t}{1-t-\cdots-t^k}\right)\left(\frac{1-\sqrt{1-4t^2}}{2t^2}, \frac{1-\sqrt{1-4t^2}}{2t}\right)\cdot\frac{1}{1-t} \\ &= \left(\frac{1}{1-t-\cdots-t^k}, \frac{t}{1-t-\cdots-t^k}\right)\cdot\frac{1}{\sqrt{1-4t^2}}\left(1+\frac{1-\sqrt{1-4t^2}}{2t}\right) \\ &= \frac{1}{\sqrt{(1-t-\cdots-t^k)^2-4t^2}}\left(1+\frac{1-t-\cdots-t^k-\sqrt{(1-t-\cdots-t^k)^2-4t^2}}{2t}\right). \end{aligned}$$

Expanding $\frac{1}{\sqrt{1-4t^2}}\left(1+\frac{1-\sqrt{1-4t^2}}{2t}\right)$ as follows

$$\frac{1}{\sqrt{1-4t^2}}+\frac{1}{\sqrt{1-4t^2}}\cdot\frac{1-\sqrt{1-4t^2}}{2t}=\sum_{j=0}^{\infty}\binom{2j}{j}t^{2j}+\sum_{j=0}^{\infty}\binom{2j+1}{j}t^{2j+1},$$

it is straightforward to obtain $S_n^{(k)}=\sum_{j=0}^{n}T_{n-j}^{(j+1)}\binom{j}{\lfloor\frac{j}{2}\rfloor}$ from the matrix equation
$S(t)=\left(\frac{1}{1-t-\cdots-t^k}, \frac{t}{1-t-\cdots-t^k}\right)\cdot\frac{1}{\sqrt{1-4t^2}}\left(1+\frac{1-\sqrt{1-4t^2}}{2t}\right)$. □

From (4.7) and Theorem 3.4, we find that the generating function for the row sums of the k-generalized left-bounded rhombus is equal to the sum of the generating functions of the first two columns of $R^{(k)}$. Hence, $S_n^{(k)}=r_{n,0}^{(k)}+r_{n,1}^{(k)}$.

Example 4.5. *For $k=2$, we have*

$$R^{(2)}=\left(\frac{1}{1-t-t^2}, \frac{t}{1-t-t^2}\right)\left(\frac{1}{\sqrt{1-4t^2}}, \frac{1-\sqrt{1-4t^2}}{2t}\right),$$

$$S^{(2)}=\left(\frac{1}{1-t-t^2}, \frac{t}{1-t-t^2}\right)\left(\frac{1-\sqrt{1-4t^2}}{2t^2}, \frac{1-\sqrt{1-4t^2}}{2t}\right).$$

Using the first six rows of these matrices, we have the matrix identities:

$$\begin{pmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 4&2&1&0&0&0\\ 9&8&3&1&0&0\\ 29&22&13&4&1&0\\ 82&72&42&19&5&1 \end{pmatrix}=\begin{pmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 2&2&1&0&0&0\\ 3&5&3&1&0&0\\ 5&10&9&4&1&0\\ 8&20&22&14&5&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 2&0&1&0&0&0\\ 0&3&0&1&0&0\\ 6&0&4&0&1&0\\ 0&10&0&5&0&1 \end{pmatrix},$$

$$\begin{pmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 3&2&1&0&0&0\\ 6&7&3&1&0&0\\ 16&18&12&4&1&0\\ 40&53&37&18&5&1 \end{pmatrix}=\begin{pmatrix} 1&0&0&0&0&0\\ 1&1&0&0&0&0\\ 2&2&1&0&0&0\\ 3&5&3&1&0&0\\ 5&10&9&4&1&0\\ 8&20&22&14&5&1 \end{pmatrix}\begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 1&0&1&0&0&0\\ 0&2&0&1&0&0\\ 2&0&3&0&1&0\\ 0&5&0&4&0&1 \end{pmatrix}.$$

Example 4.6. *For $k = 3$, we have*

$$R^{(3)} = \left(\frac{1}{1-t-t^2-t^3}, \frac{t}{1-t-t^2-t^3}\right)\left(\frac{1}{\sqrt{1-4t^2}}, \frac{1-\sqrt{1-4t^2}}{2t}\right),$$

$$S^{(3)} = \left(\frac{1}{1-t-t^2-t^3}, \frac{t}{1-t-t^2-t^3}\right)\left(\frac{1-\sqrt{1-4t^2}}{2t^2}, \frac{1-\sqrt{1-4t^2}}{2t}\right).$$

Using the first six rows of these matrices, we have the following matrix identities:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 & 0 \\ 10 & 8 & 3 & 1 & 0 & 0 \\ 31 & 24 & 13 & 4 & 1 & 0 \\ 93 & 78 & 45 & 19 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 4 & 5 & 3 & 1 & 0 & 0 \\ 7 & 12 & 9 & 4 & 1 & 0 \\ 13 & 26 & 25 & 14 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 1 & 0 & 0 \\ 6 & 0 & 4 & 0 & 1 & 0 \\ 0 & 10 & 0 & 5 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & 0 & 0 & 0 \\ 7 & 7 & 3 & 1 & 0 & 0 \\ 18 & 20 & 12 & 4 & 1 & 0 \\ 48 & 59 & 40 & 18 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 & 0 \\ 4 & 5 & 3 & 1 & 0 & 0 \\ 7 & 12 & 9 & 4 & 1 & 0 \\ 13 & 26 & 25 & 14 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 0 & 5 & 0 & 4 & 0 & 1 \end{pmatrix}.$$

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