A PRIMER FOR THE FIBONACCI NUMBERS: PART IX

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TO PROVE: F_n Divides F_{nk}

For many years, it has been known that the n^{th} Fibonacci number F_n divides F_m if and only if n divides m, $n > 2$. Many different proofs have been given; it will be instructive and entertaining to examine some of them.

Some special cases are very easy. It is obvious that F_k divides F_{2k}, for $F_{2k} = F_k L_k$. If we wish only to prove that F_n divides F_{nk} when k is a power of 2, the identity

$$F_{2^j n} = F_n L_n L_{2n} L_{4n} \cdots L_{2^{j-1} n}$$

suffices.

1. PROOFS USING THE BINET FORM

Perhaps the simplest proof to understand is one which depends upon simple algebra and the Binet form (see [1]),

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad (1)$$

where

$$\alpha = (1 + \sqrt{5})/2, \qquad \beta = (1 - \sqrt{5})/2$$

are the roots of $x^2 - x - 1 = 0$. Then

$$F_n = \frac{\alpha^{nk} - \beta^{nk}}{\alpha - \beta} = \frac{\alpha^k - \beta^k}{\alpha - \beta} \cdot (M) = F_k M,$$

where

$$M = \alpha^{(n-1)k} + \alpha^{(n-2)k}\beta^k + \alpha^{(n-3)k}\beta^{2k} + \cdots + \alpha^k \beta^{(n-2)k} + \beta^{(n-1)k} .$$

If M is an integer, then F_k divides F_{nk}, $k \neq 0$.

Since $\alpha\beta = -1$, if $(n-1)k$ is odd, pairing the first and last terms, second and next to last terms, and so on,

$$M = (\alpha^{(n-1)k} + \beta^{(n-1)k}) + (-1)^k(\alpha^{(n-3)k} + \beta^{(n-3)k}) + (-1)^{2k}(\alpha^{(n-5)k} + \beta^{(n-5)k}) + \ldots$$

$$= L_{(n-1)k} + (-1)^k L_{(n-3)k} + (-1)^{2k} L_{(n-5)k} + \ldots ,$$

where the n^{th} Lucas number is given by

(2) $$L_n = \alpha^n + \beta^n .$$

Thus, M is the sum of integers, and hence an integer. If $(n-1)k$ is even, the symmetric pairs can again be formed except for the middle term which is

$$(\alpha\beta)^{(n-1)k/2} = (-1)^{(n-1)k/2} ,$$

again making M an integer. Thus, F_k divides F_{nk}, or, F_n divides F_m if n divides m.

2. PROOFS BY MATHEMATICAL INDUCTION

Other proofs can be derived, starting with a known identity and using mathematical induction. For example, use the known identity (see [2])

(3) $$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n .$$

Let $m = nk$:

(4) $$F_{nk+n} = F_{n(k+1)} = F_{nk} F_{n+1} + F_{nk-1} F_n .$$

Cbviously, F_n divides F_n and F_n divides F_{2n}, for $F_{2n} = F_n L_n$, so that F_n divides F_{kn} for $k = 1, 2$. Assume that F_n divides F_{in} for $i = 1, 2, \ldots, k$. Then, since F_n divides F_n and F_n divides F_{kn}, identity (4) forces F_n also to divide $F_{n(k+1)}$, so that F_n divides F_{kn} for all positive integers k.

Another identity, easily proved using (2) and (3), which leads to an easy proof by mathematical induction is

(5) $$L_n F_{m-n} + F_n L_{m-n} = 2F_m \text{ .}$$

Let m = nk, yielding

(6) $$L_n F_{n(k-1)} + F_n L_{n(k-1)} = 2F_{nk} \text{ .}$$

If F_n divides F_n and F_n divides $F_{n(k-1)}$, then F_n must divide F_{nk}, for $|F_n| > 2$.

A less obvious identity given by Siler [3] also yields a proof by mathematical induction:

(7) $$((-1)^n + 1 - L_n)\left(\sum_{i=1}^{k} F_{in}\right) = (-1)^n F_{kn} - F_{n(k+1)} + F_n \text{ .}$$

If F_n divides F_{in} for i = 1, 2, 3, ..., k, then F_n is a factor of the left-hand member of (7). Since F_n divides F_n and F_n divides F_{kn}, F_n must also divide $F_{n(k+1)}$, so that F_n divides F_{kn} for all positive integers k.

3. PROOFS FROM GENERATING FUNCTIONS AND POLYNOMIALS

Now let us look for elegance. Suppose that we have proved the generating function identity given in [4],

$$\frac{F_n x}{1 - L_n x + (-1)^n x^2} = \sum_{k=0}^{\infty} F_{nk} x^k \text{ .}$$

Then, since the leading coefficient of the divisor is one and the resulting operations of division are multiplying, adding, and subtracting integers, the quotient coefficients F_{nk}/F_n of powers of x are integers, and F_n divides F_{nk} for all integers $k \geq 0$.

Let us develop a generating function for a related proof that L_n divides L_{kn} whenever k is odd. Applying (2) and the formula for summing an infinite geometric progression,

$$\sum_{i=0}^{\infty} L_{(2i+1)n} x^i = \sum_{i=0}^{\infty} \alpha^{n(2i+1)} x^i + \sum_{i=0}^{\infty} \beta^{n(2i+1)} x^i$$

$$= \frac{\alpha^n}{1 - \alpha^{2n} x} + \frac{\beta^n}{1 - \beta^{2n} x}$$

$$= \frac{(\alpha^n + \beta^n)(1 - (-1)^n x)}{1 - (\alpha^{2n} + \beta^{2n})x + (\alpha\beta)^{2n} x^2}$$

$$= \frac{L_n(1 - (-1)^n x)}{1 - L_{2n} x + x^2} \quad .$$

Then

$$\sum_{i=0}^{\infty} \frac{L_{(2i+1)n}}{L_n} x^i = \frac{1 - (-1)^n x}{1 - L_{2n} x + x^2} ,$$

so that by the same reasoning given for the Fibonacci generating function above, $L_{(2i+1)n}/L_n$ is an integer.

Next, we prove that $L_{(2k+1)n}/L_n$ is an integer another way. Now it is true that

$$L_{(2k+1)n} = L_n L_{2kn} - (-1)^{n+1} L_{(2k-1)n}$$

so that

$$\frac{L_{(2k+1)n}}{L_n} = L_{2n} - (-1)^{n+1} \frac{L_{(2k-1)n}}{L_n} \quad .$$

Thus, we are set up to use mathematical induction since when $k = 1$, it is clear that L_n divides L_n. Thus, if $L_{(2k-1)n}/L_n$ is an integer, then $L_{(2k+1)n}/L_n$ is also an integer. The proof is complete by mathematical induction.

We can carry this one step further, and prove that L_m is not divisible by L_n if $m \neq (2k+1)n$, $n \geq 2$, for

$$L_{(2k+1)n+j} = L_n L_{2kn+j} + (-1)^n L_{(2k-1)n+j}, \qquad j = 1, 2, 3, \ldots, 2n-1 .$$

Thus, given that some $j = 1, 2, 3, \ldots,$ or $2n - 1$ exists so that $L_{(2k+1)n+j}$ is divisible by L_n, then by the method of infinite descent, $L_{(2k-1)n+j}$ is divisible by L_n for this same $j = 1, 2, 3, \ldots,$ or $2n - 1$. This will ultimately yield the inequality

$$-|L_n| < L_{-n+j} < L_n \quad ,$$

which is clearly a contradiction since the L_s in that range are all smaller than L_n, $n \geq 2$. The same technique can be used on F_{nk} and F_k to prove that F_n divides F_m only if n divides m, $n > 2$. (Since $F_2 = 1$ divides all F_n, we must make the qualification $n > 2$.)

If the theory of Fibonacci polynomials is at our disposal, the theorem that F_n divides F_m if and only if n divides m, $n > 2$, becomes a special case. (See [5]).

If the following identity is accepted (proved in [5]),

$$F_m = F_n\left(\sum_{i=0}^{p-1} (-1)^{in} L_{m-(2i+1)n}\right) + (-1)^{pn} F_{m-2pn} , \qquad p \geq 1,$$

when $|n| < |m|$, $n \neq 0$, the identity can be interpreted in terms of quotients and remainders; the quotient being a sum of Lucas numbers and the remainder of least absolute value being a Fibonacci number or its negative. The remainder is zero if and only if either $F_{m-2pn} = 0$ or $F_{m-2pn} = \pm F_n$, in which case the quotient is changed by ± 1. In the first case, $m - 2pn = 0$, so that m is an even multiple of n; and in the second, $m - 2pn = \pm n$, with m an odd multiple of n. So, F_n divides F_m if and only if n divides m, $n > 2$.

That F_n divides F_m only if n divides m can also be proved through use of the Euclidean Algorithm [2] or as the solution to a Diophantine equation [6] to establish that

$$(F_m, F_n) = F_{(m,\, n)} \qquad (m \geq n > 2),$$

or, that the greatest common divisor of two Fibonacci numbers is a Fibonacci number whose subscript is the greatest common divisor of the subscripts of the other two Fibonacci numbers.

4. THE GENERAL CASE

A second proof that L_n divides L_m if and only if $m = (2k+1)n$, $n \geq 2$, provides a springboard for studying the general case. The identity

$$L_{m+n} = F_{m+1}L_n + F_mL_{n-1} \tag{8}$$

indicates that L_n divides L_{m+n} if L_n divides F_m. Since $F_{2p} = L_pF_p$, L_p divides F_{2p}. But since

$$F_{2(k+1)p} = F_{2kp+2p} = F_{2kp}F_{2p+1} + F_{2kp-1}F_{2p} \quad ,$$

whenever L_p divides F_{2kp}, it must divide $F_{2(k+1)p}$, and we have proved by mathematical induction that L_p divides F_{2kp} for all positive integers k. Then, returning to (8), if $m = 2kn$, L_n divides L_{m+n}, or,

$$L_{2kn+n} = L_{(2k+1)n} = F_{2kn+1}L_n + F_{2kn}L_{n-1} \ ,$$

so that L_n divides $L_{(2k+1)n}$.

To prove that L_n divides L_m only if $m = (2k + 1)n$, $n \geq 2$, we prove that L_n divides F_m only if $m = 2kn$, $n \geq 2$. We use the identity

$$F_{2n-j} = L_nF_{n-j} + (-1)^{n+1}F_{-j} \ , \qquad j = 1, 2, \ldots, n-1 \quad ,$$

to show that L_n cannot divide F_{2n-j}. If L_n divides F_{2n-j}, then L_n must divide F_{-j}, but $L_n > F_n > |F_{-j}|$, clearly a contradiction. Thus, L_n divides L_m if and only if $m = (2k+1)n$. A proof of this same theorem using algebraic numbers is given by Carlitz in [7].

Now we consider the general case. Given a Fibonacci sequence defined by

$$H_1 = p, \qquad H_2 = q, \qquad H_{n+2} = H_{n+1} + H_n \quad ,$$

under what circumstances does H_n divide H_m?

Studying a sequence such as

1, 4, 5, 9, 14, 23, 37, 60, 97, 157, 254, 411, 665, 1076, ...

quickly convinces one that each member divides other members of the sequence in a regular fashion. For example, 5 divides itself and every fifth member thereafter, while 4 divides itself and every sixth member thereafter.

The mystery is resolved by the identity

$$H_{m+n} = F_{m+1}H_n + F_mH_{n-1} .$$

If H_n divides F_m, then H_n divides every m^{th} term of the sequence thereafter. Further, divisibility of terms of H_n by an arbitrary integer p can be predicted using tables of Fibonacci entry points. If H_k is divisible by p, then H_{k+e} is the next member of the sequence divisible by p, where e is the entry point of p for the Fibonacci sequence. For example, if 41 divides H_n, then 41 divides H_{n+20} and 41 divides H_{n+20k} since 20 is the subscript of the first Fibonacci number divisible by 41, but 41 will divide no member of the sequence between H_n and H_{n+20}.

While any member of the Lucas sequence divides some Fibonacci number and hence many Fibonacci numbers (obvious by the identity $F_{2k} = L_kF_k$), it can be proved that no Fibonacci number greater than or equal to 5 divides any Lucas number. Also, it can be proved that every integer divides some Fibonacci number, which is false for generalized Fibonacci numbers and for Lucas numbers.

REFERENCES

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4. V. E. Hoggatt, Jr., and D. A. Lind, "A Primer for the Fibonacci Numbers, Part VI." Fibonacci Quarterly, Vol. 5, No. 5, Dec., 1967, pp. 445-460.
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6. Glen Michael, "A New Proof for an Old Property," Fibonacci Quarterly, Vol. 2, No. 1, February, 1964, pp. 57-58.
7. Leonard Carlitz, "A Note on Fibonacci Numbers," Fibonacci Quarterly, Vol. 2, No. 1, February, 1964, pp. 15-28.

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That F_n divides F_{nk} also follows from the identity

$$F_{kn}/F_n = \sum_{t=1}^{[(k+1)/2]} (-1)^{(n+1)(t+1)} \binom{k-t}{t-1} L_n^{k-2t+1}$$

where [x] denotes the greatest integer function. (Problem H-172, David Englund, and Problem H-135, James E. Desmond, FQ, Dec., 1969, pp. 518-519.)