

# The Prime Number Theorem

## 1. INTRODUCTION

The prime number theorem is a statement about the density of primes. Let  $\pi(x)$  denote the number of primes less than or equal to  $x$ . Gauss is generally credited with first conjecturing that  $\pi(x)$  is asymptotically

$$\text{li}(x) = \int_2^x \frac{dt}{\log t}$$

which he arrived at after receiving a table of logarithms as a present. Tables of logs were indispensable in that century because they provided a simple way of multiplying large numbers according to the “functional equation”  $\log(xy) = \log(x) + \log(y)$ . The book included as an appendix a table of primes, intended just as a mathematical curiosity. Gauss made the connection.

More often, we see the statement in the form

$$\pi(x) \sim \frac{x}{\log x} \quad \text{where } \sim \text{ denotes that the quotient tends to 1 as } x \rightarrow \infty.$$

The equivalence of this form and that of Gauss is immediate upon repeatedly integrating  $\text{li}(x)$  by parts to get

$$\text{li}(x) = \frac{x}{\log x} + \frac{x}{(\log x)^2} + \cdots + \frac{n! x}{(\log x)^{n+1}} (1 + \epsilon(x)) \quad \text{where } \epsilon(x) \rightarrow 0 \text{ as } x \rightarrow \infty$$

## 2. OUTLINE OF THE PROOF

By elementary methods (essentially partial summation), the function  $\pi(x)$  is related to the function  $\psi(x) = \sum_{n \leq x} \Lambda(n)$  where  $\Lambda(n)$  is the (von) Mangoldt function defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ with } k \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The relationship between  $\pi(x)$  and these types of functions were known to Tchebyshev and others during the mid 19th century. It was Riemann who outlined the proof using the zeta function. Recall the identity:

$$\frac{d}{ds} [-\log(\zeta(s))] = -\frac{\zeta'(s)}{\zeta(s)} = \sum_p \sum_{m=1}^{\infty} \frac{\log p}{p^{ms}} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

from the Euler product expansion. By placing the function  $\zeta'(s)/\zeta(s)$  in the integral transform

$$I(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^s \frac{ds}{s} = \begin{cases} 0 & \text{if } 0 < x < 1, \\ 1/2 & \text{if } x = 1, \\ 1 & \text{if } x > 1 \end{cases} \quad (\text{for any real number } c > 0)$$

(which can be shown by contour integration choosing an infinite rectangle to the right or left of the line  $\Re(s) = c$  according to the convergence), we recover the function  $\psi(x)$ . That is,

$$\begin{aligned} (1) \quad \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\zeta'(s)}{\zeta(s)} x^s \frac{ds}{s} &= \frac{-1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{n=1}^{\infty} \Lambda(n) \left(\frac{x}{n}\right)^s \frac{ds}{s} = \frac{-1}{2\pi i} \sum_{n=1}^{\infty} \Lambda(n) \int_{c-i\infty}^{c+i\infty} \left(\frac{x}{n}\right)^s \frac{ds}{s} \\ (2) \quad &= \sum_{n \leq x} \Lambda(n) = \psi(x) \end{aligned}$$

Pick  $c > 1$  so that the identities according to the Dirichlet series are valid. On the other hand, we may move the line of integration to the left. As we move, we pick up poles of the integrand  $\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s}$ . Since the zeta function has a simple pole at  $s = 1$ , and  $\frac{x^s}{s}$  has its only pole at  $s = 0$ , then all of the residues other than those at  $s = 0, 1$  will come from zeros of the zeta function.

This is given explicitly by the product formula

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + 1/2 \log \pi - \frac{1}{2} \frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)} + \sum_{\rho: \zeta(\rho)=0} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right)$$

which follows from the product formula for the entire function  $\xi(s) = s(s-1)\Gamma(s/2)\pi^{-s/2}\zeta(s)$  based on the theory of functions of finite order.

Moving the line of integration in (1) all the way to the left, we find that

$$(3) \quad \psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2} \log(1-x^{-2})$$

where the last term comes from the summing up of all the trivial zeros at the even integers. This requires some careful analysis owing to the contribution of the horizontal components of this infinite rectangle. To be precise, the equality actually holds for a variant of  $\psi(x)$ , often called  $\psi_0(x)$ , which is defined so that the value of  $\psi(x)$  at its discontinuities is the average of the right- and left-hand limits.

Finally, upon obtaining a zero-free region for  $\zeta(s)$  to the right of the line  $\Re(s) = 1$  and using an explicit formula for the number of zeros in a box in the critical strip of height  $T$ , we get an estimate for the effect of the term summed over zeros  $\rho$  of the zeta function. That is, we are able to conclude that  $\psi(x) = x + O(xe^{-c(\log x)^{1/2}})$ . Upon relating  $\psi(x)$  to the series

$$\sum_{n \leq x} \frac{\Lambda(n)}{\log n} =: \pi_1(x) = \pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \dots$$

the prime number theorem follows.

### 3. EXPLICIT FORMULAE

Our goal here is to review the derivation of Riemann's product formula for the zeta function in terms of its zeros. In a later section, we will see how this leads to an asymptotic computation on the number of zeros in a bounded rectangle in the critical strip. The product formula follows from complex function theory. We begin with the entire function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s).$$

We showed in class that this is an entire function of order 1 using Stirling's formula and a simple analytic continuation for  $\zeta(s)$  to  $\Re(s) > 0$  using an integral identity. Now from the theory of functions of finite order, we know that

$$\sum_{\substack{\rho_i \\ \zeta(\rho_i)=0}} |\rho_i|^{-1-\epsilon} \quad \text{converges for any } \epsilon > 0, \text{ and diverges for } \epsilon = 0,$$

since  $\xi(s)$  has order 1. Moreover, any entire function  $f$  of order  $n$  without zeros can be written in the form  $f(z) = e^{P(z)}$  where  $P(z)$  has degree  $n$ . Putting these two observations together, we have

$$\xi(s) = e^{A+Bs} \prod_{\rho_i} (1 - s/\rho_i) e^{s/\rho_i}.$$

That is, given an entire function of order 1 with enumerated zeros, we can write it as a product over zeros multiplied by an entire function  $f(z)$  that is nowhere zero. Then checking that  $f(z)$  still has order 1 by estimating its value inside and outside circles of radius  $R_n$  for a sequence  $\{R_n\} \rightarrow \infty$ , we obtain the above.

Note that the zeros of this product are precisely the zeros of  $\zeta(s)$  in the critical strip, i.e.  $0 \leq \Re(\rho_i) \leq 1$ . After all, the Gamma function is never zero according to the Weierstrass product form and the “trivial” zeros of the zeta function cancel the poles of the Gamma function at the negative even integers.

Upon logarithmic differentiation, this becomes

$$\frac{\xi'(s)}{\xi(s)} = B + \sum_{\rho_i} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right)$$

Combining the expression for  $\xi(s)$  in terms of  $\zeta(s)$  we obtain

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{1}{2} \log \pi - \frac{1}{2} \frac{\Gamma'(s/2+1)}{\Gamma(s/2+1)} + \sum_{\rho_i} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right)$$

We remark that according to the above,

$$B = \frac{\xi'(0)}{\xi(0)} = -\frac{\xi(1)}{\xi(1)}$$

from which  $B$  can be evaluated using an integral expression for  $\zeta(s)$  and taking a limit as  $s \rightarrow 1$  to obtain  $B = \gamma/2 - 1 + 1/2 \log 4\pi$ . The following interpretation will prove more useful.

We know that  $\sum |\rho|^{-1}$  diverges according to the order 1 of  $\xi$ . However, if we group zeros  $\rho$  and  $\bar{\rho}$  together, then for  $\rho = \beta + i\gamma$  we have

$$\frac{1}{\rho} + \frac{1}{\bar{\rho}} = \frac{2\beta}{\beta^2 + \gamma^2} \leq \frac{2}{|\rho|^2}$$

from which we note that  $\sum \frac{1}{\rho}$  conditionally converges. Now applying the above expression for the logarithmic derivative of  $\xi(s)$ , and using the functional equation for  $\xi$ , we find

$$B + \sum_{\rho_i} \left( \frac{1}{1-s-\rho} + \frac{1}{\rho} \right) = -B + \sum_{\rho_i} \left( \frac{1}{s-\rho} + \frac{1}{\rho} \right)$$

So that

$$B = -2 \sum_{\gamma > 0} \frac{\beta}{\beta^2 + \gamma^2}$$

Similar expansions hold for Dirichlet  $L$ -functions and will prove useful in finding upper bounds on their growth in terms of the conductor.

#### 4. ZERO-FREE REGIONS

Looking at the form of the function  $\psi(x)$  given in (3), the main term in estimating  $\psi$  will be  $x$  provided that we can show there are no zeros for  $\zeta(s)$  lying on the line  $\Re(s)$ . Moreover, by exhibiting a zero-free region to the left of the edge of the critical strip, we improve the error term in our estimate. (Recall that we gave an analytic continuation for the zeta function as a sum of  $\frac{1}{s-1}$  and a function holomorphic for  $\Re(s) > 0$ , so no other poles contribute to the contour integration.) Knowing that zeros are symmetric about the line  $\Re(s) = 1/2$  according to the functional equation implies that the truth of the Riemann hypothesis would imply the best possible error estimates for  $\psi(x)$  and consequently  $\pi(x)$ .

The proof worth remembering that  $\zeta(s) \neq 0$  for  $\Re(s) = 1$  was given by Mertens. We reproduce it now.

According to the Euler product for  $\zeta(s)$ , if  $s = \sigma + it$  with  $\sigma > 1$ , then

$$\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-m\sigma} e^{-itm \log p}$$

which implies that

$$(4) \quad \Re \log \zeta(s) = \sum_p \sum_{m=1}^{\infty} m^{-1} p^{-m\sigma} \cos(t \log p^m)$$

We want some identity declaring the zeta function bounded away from zero for all  $t$  so we go in search of cosine identities which might lead us to such a result. Mertens chose the following clever one:

$$3 + 4 \cos \theta + \cos 2\theta = 2(1 + \cos \theta)^2 \geq 0.$$

This works nicely because, according to (4), it implies that

$$3 \log \zeta(\sigma) + 4 \Re \log \zeta(\sigma + it) + \Re \log \zeta(\sigma + 2it) \geq 0$$

which is equivalent to the inequality

$$|\zeta^3(\sigma) \zeta^4(\sigma + it) \zeta(\sigma + 2it)| \geq 1$$

again with the restriction that  $\sigma > 1$ . The conclusion stunningly follows since we know that for  $s = 1 + it$ , then  $\zeta(\sigma) = \zeta(1)$  has a simple pole (in fact, as described above, this is the only pole on this line). So then a zero at  $1 + it$  would imply that the left-hand side of the above inequality goes to 0 as  $\sigma \rightarrow 1$  (since  $\zeta(\sigma + 2it)$  is bounded as  $\sigma \rightarrow 1$ ), contradicting the inequality. The argument can even be extended, using the infinite product formula for the zeta function, to give an explicit zero-free region for the zeta function. We will, however, be satisfied with the above. One marvels at the existence of such a cosine relation which allows us to distinguish precisely the desired result about analytic behavior of the zeta function. As far as I know, it has still not been shown that there is a zero-free region of form  $\Re(s) > 1 - \delta$  for some positive  $\delta$ . All regions asymptotically approach the edge of the critical strip upon taking a limit as  $|t| \rightarrow \infty$ . Later in the quarter, we will see one of the great gems of analytic number theory, Siegel's theorem, which uses a case method to prove lower bounds for Dirichlet  $L$ -functions. The cases break down according to whether one can prove a zero-free region for such functions.

## 5. COUNTING ZEROS IN THE CRITICAL STRIP

Let  $N(T)$  be the number of zeros of  $\zeta(s)$  contained in the rectangle  $0 < \sigma < 1$ ,  $0 < t < T$ . The goal of this section is an asymptotic result for the growth in  $T$  of this quantity.

Let  $R$  be the rectangle with vertices  $2, 2 + iT, -1, -1 + iT$  (any rectangle containing only the zeros of  $\zeta(s)$  described above will do). Using facts from basic complex function theory (e.g. winding number, see Ahlfors p 131),

$$N(T) = \frac{1}{2\pi i} \int_R \frac{\xi'(s)}{\xi(s)}$$

Because we know the result is a non-negative integer, we can restrict our attention to the pure imaginary part of the integrand (owing to the  $i$  in the constant multiplier). This is just the change in the argument of  $\xi(s)$  along each straight line of the boundary according to the complex valued expression of the logarithm. Hence, as notated in Davenport Ch. 15,

$$2\pi N(T) = \Delta_R \arg \xi(s)$$

But when  $s$  is real along the base of the rectangle, there is no change in  $\arg \xi(s)$ . Moreover, exploiting the symmetry of  $\xi(s)$ , we have

$$\pi N(T) = \Delta_{R^+} \arg \xi(s) \quad \text{where } R^+ \text{ denotes the pair of lines connecting } 2 \text{ to } 2 + iT \text{ to } 1/2 + iT$$

Using a functional equation for the Gamma function, we may write

$$\xi(s) = (s-1)\pi^{-s/2}\Gamma(s/2+1)\zeta(s)$$

thereby reducing the number of components for which we need to investigate the change in the argument. Let's begin.

$$\Delta_{R^+} \arg(s-1) = \arg(-1/2 + iT) - \arg(1) = \arg(-1/2 + iT) = \pi/2 + \tan^{-1}\left(\frac{-1}{2T}\right)$$

But

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots \quad \text{so} \quad \Delta_{R^+} \arg(s-1) = \pi/2 + O(T^{-1})$$

Next

$$\Delta_{R^+} \arg \pi^{-s/2} = \Delta_{R^+} \arg e^{(-\sigma/2 - it/2) \log \pi} = \Delta_{R^+} \left(-\frac{t}{2} \log \pi\right) = -\frac{T}{2} \log \pi$$

To estimate the change in argument of the Gamma factor, we use Stirling's formula. Recall that it states

$$\log \Gamma(s) = (s-1/2) \log s - s + \frac{1}{2} \log 2\pi + O(|s|^{-1}). \quad (\text{Stirling's Formula})$$

Hence,

$$\begin{aligned} \Delta_{R^+} \arg \Gamma(s/2+1) &= \text{Im}(\log \Gamma(5/4 + iT/2)) \\ &= \text{Im}((3/4 + iT/2) \log(5/4 + iT/2) - 5/4 - iT/2 + \frac{1}{2} \log 2\pi + O(T^{-1})) \\ &= \frac{T}{2} \log \frac{T}{2} - \frac{T}{2} + \frac{3\pi}{8} + O(T^{-1}) \end{aligned}$$

since

$$\log(5/4 + iT/2) = \text{Log}(\sqrt{T^2/4 + 25/16}) + i \arg(iT/2 + 5/4) = \log T + i\left(\frac{\pi}{2} + O(T^{-1})\right).$$

Upon putting these estimates together and dividing by  $\pi$ , we find

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1})$$

where  $\pi S(T) = \Delta_{R^+} \zeta(s)$  is the last remaining piece to estimate. We claim that  $S(T) = O(\log T)$ .

## 6. LAST THINGS TO DO

In the next couple of days, I want to add two more sections to this note. I want to show how you get from the function  $\psi$  to  $\Lambda$  to  $\pi$ . Then we'll consider the prime number theorem done.