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A counterexample to a 1961 "theorem" in homological algebra

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Abstract. In 1961, Jan-Erik Roos published a "theorem", which says that in an [AB4*] abelian category, $\lim^1$ vanishes on Mittag–Leffler sequences. See Propositions 1 and 5 in [4]. This is a "theorem" that many people since have known and used. In this article, we outline a counterexample. We construct some strange abelian categories, which are perhaps of some independent interest.

These abelian categories come up naturally in the study of triangulated categories. A much fuller discussion may be found in [3]. Here we provide a brief, self contained, non–technical account. The idea is to make the counterexample easy to read for all the people who have used the result in their work.

In the appendix, Deligne gives another way to look at the counterexample.

0. Introduction

Abelian categories are old, venerable objects in mathematics, playing an important rôle. We are very accustomed to working with examples, such as categories of modules over a ring R, or more generally categories of sheaves of modules. Much of our intuition comes from these examples.

Here we will see an amusing construction of new and very different abelian categories. Let me explain the phenomenon we will observe.

Suppose $\mathcal{A}$ is an abelian category, satisfying [AB3]. That is, all small coproducts (and hence all small direct limits) exist in $\mathcal{A}$. Suppose further that the category $\mathcal{A}$ has enough injectives. Let us be given a sequence of monomorphisms in $\mathcal{A}$

$$a_0 \longrightarrow a_1 \longrightarrow a_2 \longrightarrow \cdots$$

Because the category $\mathcal{A}$ has enough injectives, there is an injective object i and an embedding $a_0 \longrightarrow i$. Because i is injective and the map $a_0 \longrightarrow a_1$ is a monomorphism, the map $a_0 \longrightarrow i$ factors as

$$a_0 \longrightarrow a_1 \longrightarrow i.$$

Because $a_1 \longrightarrow a_2$ is a monomorphism, the map $a_1 \longrightarrow i$ factors as

$$a_1 \longrightarrow a_2 \longrightarrow i.$$

Proceeding by induction, we obtain for every n a map $a_n \longrightarrow i$, which combine to give a map

$$\underrightarrow{\operatorname{colim}}\{a_n\} \longrightarrow i.$$

In other words, the monomorphism $a_0 \longrightarrow i$ factors as

$$a_0 \longrightarrow \underrightarrow{\operatorname{colim}}\{a_n\} \longrightarrow i;$$

it follows that the map $a_0 \longrightarrow \underrightarrow{\operatorname{colim}}\{a_n\}$ is a monomorphism.

Thus, in an abelian category with enough injectives, given any sequence of monomorphisms

$$a_0 \longrightarrow a_1 \longrightarrow a_2 \longrightarrow \cdots$$

then the map $a_0 \longrightarrow \underrightarrow{\operatorname{colim}}\{a_n\}$ is a monomorphism.

Of course, even in the absence of injectives, this still often happens. For example, if $\mathcal{A}$ satisfies [AB5] (that is, filtered direct limits are exact), then the sequence of monomorphisms

$$\begin{array}{ccccccc} a_0 & \longrightarrow & a_0 & \longrightarrow & a_0 & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ a_0 & \longrightarrow & a_1 & \longrightarrow & a_2 & \longrightarrow & \cdots \end{array}$$

has a monomorphism for its direct limit; hence $a_0 \longrightarrow \underrightarrow{\operatorname{colim}}\{a_n\}$ is a monomorphism.

In this article, we will construct new and unusual abelian categories. In particular, we will construct an abelian category $\mathcal{A}$ satisfying [AB4] and [AB4*] (that is, coproducts and products exist and are exact), but in $\mathcal{A}$ we will construct a sequence of monomorphisms

$$a_0 \longrightarrow a_1 \longrightarrow a_2 \longrightarrow \cdots$$

so that $\underrightarrow{\operatorname{colim}}\{a_n\} = 0$.

To realise just how counterintuitive this example is, the reader should check Proposition 5 in [4], or Lemma 1.15 on page 213 of [2]. What we have here amounts to a counterexample to the Proposition stated there. This point is discussed, in infinitely more detail, in Sect. A.5 of [3].

The abelian categories we produce and study are genuinely strange. They are not categories of sheaves on any site; they are not Grothendieck abelian categories, and neither are their duals. In the book [3], they are studied for the information they provide, in the study of triangulated categories. But since they are of some independent interest, and might well arise elsewhere, the author thought this brief note might be worthwhile. It contains a very brief sketch of some of the properties that make these unusual abelian categories interesting.

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1. The construction

We begin with two definitions.

Definition 1.1. *Let $\mathcal{S}$ be an essentially small additive category. The category $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$ is defined to be the category of all additive functors*

$$F : \mathcal{S}^{op} \longrightarrow \mathcal{A}b.$$

So far, we have done nothing unusual. The category $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$ is an old friend, which can be expressed as a category of sheaves on a suitable site. In fact, it is very nearly a category of modules over a ring.

Definition 1.2. *Let α be an infinite cardinal. Let $\mathcal{S}$ be an essentially small additive category, closed under the formation of coproducts of $\leq \alpha$ of its objects.*

The category $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b) \subset \mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$ is defined to be the full subcategory of all additive functors F, which take coproducts [of $\leq \alpha$ objects in $\mathcal{S}$] to products of abelian groups.

Let us remind the reader what Definition 1.2 means. Let Λ be a set of cardinality $\leq \alpha$. Suppose we are given a family of objects in $\mathcal{S}$, of the form $\{s_\lambda, \lambda \in \Lambda\}$. By the hypothesis on $\mathcal{S}$, the coproduct exists in $\mathcal{S}$; there is a coproduct $\coprod_{\lambda\in\Lambda} s_\lambda$. For each $\lambda \in \Lambda$, the contravariant functor F gives a map

$$F\left\{\coprod_{\lambda\in\Lambda} s_\lambda\right\} \longrightarrow F(s_\lambda).$$

The universal property of the product assembles these to a map

$$F\left\{\coprod_{\lambda\in\Lambda} s_\lambda\right\} \longrightarrow \prod_{\lambda\in\Lambda} F(s_\lambda).$$

The hypothesis on F is that all such maps are isomorphisms.

Lemma 1.3. *Let α be an infinite cardinal. Suppose $\mathcal{S}$ is an essentially small additive category, containing coproducts for any collection of $\le \alpha$ of its objects. The category $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ is an abelian subcategory of $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$. That is, $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ is an abelian category, and the inclusion*

$$\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b) \qquad \subset \qquad \mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$$

is an exact functor.

Proof. Suppose $F \longrightarrow F'$ is a morphism in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. That is, F and F' are functors $\mathcal{S}^{op} \longrightarrow \mathcal{A}b$ taking coproducts of fewer than α objects to products, and $F \longrightarrow F'$ is a natural transformation. We need to show that the kernel and cokernel of the natural transformation, which are clearly objects of the big category $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$, actually lie in the subcategory $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$.

Complete the map $F \longrightarrow F'$ to an exact sequence in $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$

$$0 \longrightarrow K \longrightarrow F \longrightarrow F' \longrightarrow Q \longrightarrow 0.$$

Let $\{s_\lambda, \lambda \in \Lambda\}$ be a set of $\le \alpha$ objects in $\mathcal{S}$. Because F and F' lie in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$, the natural maps

$$F\left(\coprod_{\lambda\in\Lambda} s_\lambda\right) \longrightarrow \prod_{\lambda\in\Lambda} F(s_\lambda)$$

$$F'\left(\coprod_{\lambda\in\Lambda} s_\lambda\right) \longrightarrow \prod_{\lambda\in\Lambda} F'(s_\lambda)$$

are both isomorphisms. We deduce that in the commutative square

$$\begin{array}{ccc}
F\left(\coprod_{\lambda\in\Lambda} s_\lambda\right) & \longrightarrow & F'\left(\coprod_{\lambda\in\Lambda} s_\lambda\right) \\
\downarrow \wr & & \downarrow \wr \\
\prod_{\lambda\in\Lambda} F(s_\lambda) & \longrightarrow & \prod_{\lambda\in\Lambda} F'(s_\lambda)
\end{array}$$

the vertical maps are both isomorphisms. But $\mathcal{A}b$ satisfies [AB4*]. Hence the product of the exact sequences

$$0 \longrightarrow K(s_\lambda) \longrightarrow F(s_\lambda) \longrightarrow F'(s_\lambda) \longrightarrow Q(s_\lambda) \longrightarrow 0$$

over $\lambda \in \Lambda$ is an exact sequence. In the comparison map

$$\begin{array}{ccccccc}
K\left(\coprod_{\lambda\in\Lambda} s_\lambda\right) & \longrightarrow & F\left(\coprod_{\lambda\in\Lambda} s_\lambda\right) & \longrightarrow & F'\left(\coprod_{\lambda\in\Lambda} s_\lambda\right) & \longrightarrow & Q\left(\coprod_{\lambda\in\Lambda} s_\lambda\right) \\
\downarrow & & \downarrow \wr & & \downarrow \wr & & \downarrow \\
\prod_{\lambda\in\Lambda} K(s_\lambda) & \longrightarrow & \prod_{\lambda\in\Lambda} F(s_\lambda) & \longrightarrow & \prod_{\lambda\in\Lambda} F'(s_\lambda) & \longrightarrow & \prod_{\lambda\in\Lambda} Q(s_\lambda)
\end{array}$$

both the top and bottom rows are exact. It easily follows that the natural maps

$$K\left(\coprod_{\lambda\in\Lambda} s_\lambda\right) \longrightarrow \prod_{\lambda\in\Lambda} K(s_\lambda)$$

$$Q\left(\coprod_{\lambda\in\Lambda} s_\lambda\right) \longrightarrow \prod_{\lambda\in\Lambda} Q(s_\lambda)$$

are both isomorphisms. □

This completes the construction. Out of any essentially small additive category $\mathcal{S}$, closed under coproducts of $\leq \alpha$ objects, we have produced an abelian category $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$.

2. Properties of the construction

It is now incumbent on us to study the properties of this construction. We begin with the easy and the well–known.

Lemma 2.1. *Let α be an infinite cardinal. Let $\mathcal{S}$ be an essentially small additive category, closed under coproducts of $\leq \alpha$ of its objects. Then the abelian category $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ satisfies [AB4*]; it contains arbitrary small products, and products are exact.*

Proof. Since products of functors taking coproducts to products also take coproducts to products, the product, in $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$, of a family of objects in the smaller

$$\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b) \qquad \subset \qquad \mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$$

lies in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. It follows that not only do products exist in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$; the inclusion into $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$ preserves them. Since products are exact in $\mathcal{A}b$, they are exact in $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$, and hence also in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. □

Notation 2.2. Next we will want to study the existence and exactness of colimits in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. The colimits that obviously exist are the α–filtered colimits. Recall that a category I is called α–filtered if every subcategory $I' \subset I$ of cardinality $\leq \alpha$ can be embedded in a subcategory $I' \subset I'' \subset I$, where I'' has a terminal object. The colimit of any functor $I \longrightarrow \mathcal{A}$, where I is α–filtered, is called an α–filtered colimit. It is very classical that α–filtered colimits, taken in $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$, of functors

$$I \longrightarrow \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$$

actually lie in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. The colimits therefore exist in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$, and agree with the colimits in $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$.

In this section, we will consider colimits both in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ and in $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$. This being the case, we need to adopt some notational conventions to stop us from getting confused. In this section, when we write $\underrightarrow{\mathrm{colim}} F_\mu$, then we assume that the F_μ form an α–filtered system. If all the F_μ's lie in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$, then it does not matter whether the colimit is being computed in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ or in $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$. The only colimits we will consider in this section which are not α–filtered are coproducts. Coproducts in $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$ (resp. in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$) will be denoted

$$\bigoplus F_\mu, \qquad \text{respectively} \qquad \coprod F_\mu.$$

Note that we do not yet know that the coproduct on the right exists.

Remark 2.3. Since $\mathcal{S}$ is an additive category, the representable functors $\mathcal{S}(-, s)$ are additive. And representable functors always take coproducts to products. Therefore, all the functors $\mathcal{S}(-, s)$ are objects in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. The Yoneda map, which is usually written as a functor $\mathcal{S} \longrightarrow \mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$, can be factored

$$\mathcal{S} \longrightarrow \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b) \qquad \subset \qquad \mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b).$$

It is classical that $\mathcal{S}(-, s)$ is a projective object in the large category $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$. It must therefore also be projective in the exact subcategory $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. It is also a well–known consequence of Yoneda's lemma that every object in $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$ is the quotient of a direct sum of representables $\mathcal{S}(-, s)$. We will next prove

Lemma 2.4. *Let α be an infinite cardinal. Let $\mathcal{S}$ be an essentially small additive category, closed under coproducts of $\leq \alpha$ of its objects. Then the Yoneda map, $\mathcal{S} \longrightarrow \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$, preserves coproducts of $\leq \alpha$ objects.*

Proof. Suppose $\{s_\lambda, \lambda \in \Lambda\}$ is a family of $\leq \alpha$ objects in $\mathcal{S}$. Suppose that F is an object in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$, and suppose that, for each $\lambda \in \Lambda$, we are given a morhpism

$$\mathcal{S}(-, s_\lambda) \longrightarrow F.$$

By Yoneda's lemma, each of the given maps $\mathcal{S}(-, s_\lambda) \longrightarrow F$ corresponds, uniquely, to an element $r_\lambda \in F(s_\lambda)$. This gives us an element,

$$\prod_{\lambda\in\Lambda} r_\lambda \quad \in \quad \prod_{\lambda\in\Lambda} F(s_\lambda) \quad = \quad F\left(\coprod_{\lambda\in\Lambda} s_\lambda\right).$$

The last equality, $\prod_{\lambda\in\Lambda} F(s_\lambda) = F\left(\coprod_{\lambda\in\Lambda} s_\lambda\right)$, is because $F \in \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$.

Now applying Yoneda's lemma again, the above corresponds to a unique map

$$\mathcal{S}\left(-, \coprod_{\lambda\in\Lambda} s_\lambda\right) \longrightarrow F.$$

In other words, any collection of $\leq \alpha$ maps in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$

$$\mathcal{S}(-, s_\lambda) \longrightarrow F$$

factors through a unique map

$$\mathcal{S}\left(-, \coprod_{\lambda\in\Lambda} s_\lambda\right) \longrightarrow F.$$

This precisely says that

$$\mathcal{S}\left(-, \coprod_{\lambda\in\Lambda} s_\lambda\right) \quad = \quad \coprod_{\lambda\in\Lambda} \mathcal{S}(-, s_\lambda).$$

□

Proposition 2.5. *Let α be an infinite cardinal. Let $\mathcal{S}$ be an essentially small additive category, closed under coproducts of $\leq \alpha$ of its objects. Then the inclusion $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b) \subset \mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$ has a left adjoint.*

Proof. [In the interest of keeping this paper self contained, we give the entire proof. It is a modification of the argument of Gabriel and Ulmer; see Korollar 5.8 on page 60 of [1].]

We want to produce a functor

$$L : \mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b) \longrightarrow \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b),$$

left adjoint to the inclusion. That is, for every object $F \in \mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$, we wish to produce an object $LF \in \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$, so that for any $G \in \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$,

$$\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)\{F, G\} \quad = \quad \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)\{LF, G\}.$$

Let us begin by treating the special case where

$$F(-) = \bigoplus_{i\in I} \mathcal{S}(-, s_i)$$

is the direct sum of representables (see Remark 2.3). In this special case, we define

$$LF(-) = \operatorname*{colim}_{I' \subset I,\ \#I' \leq \alpha} \mathcal{S}\left(-, \coprod_{i \in I'} s_i\right).$$

The notation means that we take the colimit over all $I' \subset I$, where the cardinality of I' is $\leq \alpha$, and hence $\coprod_{i \in I'} s_i$ makes sense. Notice that the right hand side is an α–filtered colimit of objects in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$, and hence lies in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. By Lemma 2.4, $\mathcal{S}\left(-, \coprod_{i \in I'} s_i\right)$ is the coproduct in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ of the objects $\mathcal{S}(-, s_i)$. Given any object $G \in \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$, to give a map

$$\mathcal{S}\left(-, \coprod_{i \in I'} s_i\right) \longrightarrow G(-)$$

is just to give, for every $i \in I'$, maps $\mathcal{S}(-, s_i) \longrightarrow G$. Putting this together, we have that for any $G \in \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$

$$\begin{aligned}
\operatorname{Hom}(LF, G) &= \lim_{I' \subset I,\ \#I' \leq \alpha} \operatorname{Hom}\left\{\mathcal{S}\left(-, \coprod_{i \in I'} s_i\right), G\right\} \\
&= \lim_{I' \subset I,\ \#I' \leq \alpha} \prod_{i \in I'} \operatorname{Hom}\{\mathcal{S}(-, s_i), G\} \\
&= \prod_{i \in I} \operatorname{Hom}\{\mathcal{S}(-, s_i), G\} \\
&= \operatorname{Hom}(F, G).
\end{aligned}$$

That is, LF satisfies the required universal property.

But now note that, by Remark 2.3, every object $F \in \mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$ has a projective presentation

$$F'' \longrightarrow F' \longrightarrow F \longrightarrow 0,$$

with

$$F''(-) = \bigoplus_{i \in I} \mathcal{S}(-, s_i), \qquad F'(-) = \bigoplus_{j \in J} \mathcal{S}(-, s_j).$$

Define LF to be the quotient in the exact sequence

$$LF'' \longrightarrow LF' \longrightarrow LF \longrightarrow 0,$$

with LF'' and LF' as above. For any object $G \in \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
0 \longrightarrow & \operatorname{Hom}(LF, G) & \longrightarrow & \operatorname{Hom}(LF', G) & \longrightarrow & \operatorname{Hom}(LF'', G) \\
& & & \wr\downarrow & & \wr\downarrow \\
0 \longrightarrow & \operatorname{Hom}(F, G) & \longrightarrow & \operatorname{Hom}(F', G) & \longrightarrow & \operatorname{Hom}(F'', G)
\end{array}$$

from which it immediately follows that $\mathrm{Hom}(LF, G)$ is naturally isomorphic to $\mathrm{Hom}(F, G)$. □

Corollary 2.6. *Let α be an infinite cardinal. Let $\mathcal{S}$ be an essentially small additive category, closed under coproducts of $\leq \alpha$ of its objects. Then the category $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ satisfies [AB3]. It contains arbitrary small coproducts of its objects.*

Proof. Let L be the left adjoint to the inclusion

$$\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b) \qquad \subset \qquad \mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b).$$

The existence of L is guaranteed by Proposition 2.5. Let $\{F_\mu, \mu \in M\}$ be a family of objects in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. Because coproducts exist in $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$, there is a coproduct in $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$. We denote it

$$\bigoplus_{\mu \in M} F_\mu .$$

The functor L is a left adjoint, and hence preserves coproducts. It follows that

$$L\left\{\bigoplus_{\mu \in M} F_\mu\right\}$$

is the coproduct in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$, of $LF_\mu = F_\mu$. □

Now make the following definition

Definition 2.7. *Let α be an infinite cardinal. Let $\mathcal{S}$ be an essentially small additive category, closed under coproducts of $\leq \alpha$ of its objects. Let $s \longrightarrow s' \longrightarrow s''$ be two morphisms in $\mathcal{S}$, whose composite is zero. The sequence is called* exact *if it induces an exact sequence in the abelian category $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. That is, if the sequence of functors*

$$\mathcal{S}(-, s) \longrightarrow \mathcal{S}(-, s') \longrightarrow \mathcal{S}(-, s'')$$

is exact.

Remark 2.8. Suppose the category $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ satisfies [AB4]; that is, coproducts are exact. Then it would follow in particular that if we take a family of $\leq \alpha$ exact sequences in $\mathcal{S}$, of the form

$$s_\lambda \longrightarrow s'_\lambda \longrightarrow s''_\lambda,$$

then the sequence

$$\coprod_{\lambda \in \Lambda} s_\lambda \longrightarrow \coprod_{\lambda \in \Lambda} s'_\lambda \longrightarrow \coprod_{\lambda \in \Lambda} s''_\lambda$$

is also exact in $\mathcal{S}$. After all, by Lemma 2.4, coproducts in $\mathcal{S}$ agree with those in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$.

There is therefore a necessary condition for the category $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ to satisfy [AB4]. If all coproducts are to be exact in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$, then at the very least the coproducts of $\leq \alpha$ objects must be exact in $\mathcal{S}$. The next Proposition asserts that, under reasonable hypotheses, this necessary condition is also sufficient.

Proposition 2.9. *Let α be an infinite cardinal. Let $\mathcal{S}$ be an essentially small additive category, closed under coproducts of $\leq \alpha$ of its objects. Suppose the following two conditions hold*

2.9.1. *Any morphism $s' \longrightarrow s''$ in $\mathcal{S}$ may be completed to an exact sequence $s \longrightarrow s' \longrightarrow s''$.*

2.9.2. *The coproduct of any collection of $\leq \alpha$ exact sequences in $\mathcal{S}$ is exact in $\mathcal{S}$.*

Then the category $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ satisfies [AB4].

Proof. Consider the following full subcategory $A(\mathcal{S}) \subset \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. An object $F \in A(\mathcal{S})$ is any functor $F : \mathcal{S}^{op} \longrightarrow \mathcal{A}b$ admitting a presentation

$$\mathcal{S}(-, s) \longrightarrow \mathcal{S}(-, t) \longrightarrow F(-) \longrightarrow 0.$$

It is well–known that under Hypothesis 2.9.1 above, this is an abelian subcategory of $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$; the proof is basically the same as the proof showing that coherent sheaves on a noetherian scheme form an abelian category. The objects $F \in A(\mathcal{S})$ will be referred to as "coherent functors".

Next observe that any object of $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ is an α–filtered colimit of coherent functors. Let F be an object of $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. It admits a projective presentation in $\mathcal{C}at(\mathcal{S}^{op}, \mathcal{A}b)$

$$\bigoplus_{i\in I} \mathcal{S}(-, s_i) \longrightarrow \bigoplus_{j\in J} \mathcal{S}(-, s_j) \longrightarrow F(-) \longrightarrow 0.$$

But then F is the α–filtered colimit of all quotients

$$\bigoplus_{i\in I'} \mathcal{S}(-, s_i) \longrightarrow \bigoplus_{j\in J'} \mathcal{S}(-, s_j) \longrightarrow F'(-) \longrightarrow 0,$$

where the cardinalities of $I' \subset I$ and of $J' \subset J$ are bounded by α. Now the functor L has a right adjoint, and preserves colimits. Hence $LF = F$ is the α–filtered colimit of LF', and LF' has a presentation

$$\mathcal{S}\left(-, \coprod_{i\in I'} s_i\right) \longrightarrow \mathcal{S}\left(-, \coprod_{j\in J'} s_j\right) \longrightarrow LF'(-) \longrightarrow 0.$$

A similar argument shows that any short exact sequence

$$0 \longrightarrow F \longrightarrow G \longrightarrow H \longrightarrow 0$$

in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ is an α–filtered colimit of short exact sequences in $A(\mathcal{S})$; we leave the details to the reader. Hence to prove that coproducts of short exact sequences in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ exact, it suffices to assume that all the sequences for which we take the coproduct lie in $A(\mathcal{S}) \subset \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. Furthermore, a coproduct over a set Λ is the α–filtered colimit of coproducts over subsets $\Lambda' \subset \Lambda$, of cardinality $\leq \alpha$. We may therefore assume the coproduct is over an index set Λ of cardinality $\leq \alpha$.

Assume therefore that Λ is a set of cardinality $\leq \alpha$, and for every $\lambda \in \Lambda$ we have a short exact sequence in $A(\mathcal{S})$

$$0 \longrightarrow F_\lambda \longrightarrow G_\lambda \longrightarrow H_\lambda \longrightarrow 0.$$

Because F_λ and H_λ lie in $A(\mathcal{S})$, each admits a presentation

$$\begin{array}{ccccccc} \mathcal{S}(-, f_1^\lambda) & \longrightarrow & \mathcal{S}(-, f_0^\lambda) & \longrightarrow & F_\lambda & \longrightarrow & 0, \\ \mathcal{S}(-, h_1^\lambda) & \longrightarrow & \mathcal{S}(-, h_0^\lambda) & \longrightarrow & H_\lambda & \longrightarrow & 0. \end{array}$$

By 2.9.1, these presentations may be continued to a projective resolution

$$\begin{array}{ccccccccc} \longrightarrow & \mathcal{S}(-, f_2^\lambda) & \longrightarrow & \mathcal{S}(-, f_1^\lambda) & \longrightarrow & \mathcal{S}(-, f_0^\lambda) & \longrightarrow & F_\lambda & \longrightarrow 0, \\ \longrightarrow & \mathcal{S}(-, h_2^\lambda) & \longrightarrow & \mathcal{S}(-, h_1^\lambda) & \longrightarrow & \mathcal{S}(-, h_0^\lambda) & \longrightarrow & H_\lambda & \longrightarrow 0. \end{array}$$

By standard homological algebra, these may be combined to give a projective resolution of the short exact sequence

$$0 \longrightarrow F_\lambda \longrightarrow G_\lambda \longrightarrow H_\lambda \longrightarrow 0.$$

We want to show that the coproduct of these $\leq \alpha$ short exact sequences is exact. It suffices to show that the coproduct in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ of their resolutions is a resolution of a short exact sequence. But this is immediate from 2.9.2.

[This trick, of reducing statements about $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ to α–filtered direct limits of statements about representables, is very much in the spirit of Gabriel and Ulmer [1]. See, for example, the proof of Satz 5.9 on page 60 of [1]. A completely different proof of [AB4], somewhat more explicit, may be found in [3], Lemma 6.3.2.] □

3. A counterexample

In this section, we look at a special case of a category $\mathcal{S}$. We let our cardinal α be $\aleph_0$, the first infinite cardinal. Now we define the category $\mathcal{S}$.

Definition 3.1. *Let $\mathcal{S}$ be the category whose objects are complete, non–archimedean, normed abelian groups of cardinality $\leq 2^{\aleph_0}$, and whose morphisms are the contractions.*

Remark 3.2. Definition 3.1 is quite a mouthful, so let us paraphrase it. An object of $\mathcal{S}$ is an abelian group A of cardinality $\leq 2^{\aleph_0}$, having a norm map. That is

3.2.1. *For every $a \in A$, there is a number $\|a\| \in \mathbb{R}$. These numbers satisfy the inequality $\|a\| \geq 0$, with equality if and only if $a = 0$.*

3.2.2. *The norm is non–archimedean. It satisfies the inequality*

$$\|a - b\| \leq max(\|a\|, \|b\|).$$

3.2.3. *The group A is complete with respect to the metric induced by the norm.*

The morphisms in the category $\mathcal{S}$ are the contractions. They are homomorphisms of abelian groups $f : A \longrightarrow B$ satisfying

$$\| f(a)\| \leq \|a\|.$$

Lemma 3.3. *The category $\mathcal{S}$ contains coproducts of $\leq \aleph_0$ of its objects.*

Proof. Suppose we are given $\leq \aleph_0$ objects of $\mathcal{S}$, that is countably many objects $\{A_0, A_1, A_2, \cdots\}$. The A_i are all abelian groups of cardinality $\leq 2^{\aleph_0}$. Therefore the set theoretic product group

$$\prod_{i=0}^{\infty} A_i$$

has cardinality

$$\leq \left\{2^{\aleph_0}\right\}^{\aleph_0} \quad = \quad 2^{\aleph_0 \times \aleph_0} \quad = \quad 2^{\aleph_0}.$$

Define a norm map on $\prod_{i=0}^{\infty} A_i$ by the formula

$$\left\| \prod_{i=0}^{\infty} a_i \right\| \quad = \quad \sup_{i=0}^{\infty} \|a_i\|.$$

This norm takes its value in $\mathbb{R} \cup \{\infty\}$. The coproduct of the objects A_i in the category $\mathcal{S}$ is the subset of all elements of the set theoretic product, which are sequences whose norm tends to zero. That is,

$$\coprod_{i=0}^{\infty} A_i \qquad \subset \qquad \prod_{i=0}^{\infty} A_i,$$

and the condition for a sequence $\{a_0, a_1, a_2, \cdots\} \in \prod_{i=0}^{\infty} A_i$ to lie in the smaller $\coprod_{i=0}^{\infty} A_i$ is that

$$\lim_{i\to\infty} \|a_i\| = 0.$$

We need to establish that this satisfies the universal property of the coproduct.

Suppose for each $0 \le i < \infty$ we have, in the category $\mathcal{S}$, a map $f_i : A_i \longrightarrow B$. That is, we have a contraction. Define

$$f : \coprod_{i=0}^{\infty} A_i \longrightarrow B$$

by the formula

$$f(a_0, a_1, a_2, \cdots) = \sum_{i=0}^{\infty} f_i(a_i).$$

This sum converges since as $i \to \infty$, we have first $\|a_i\| \to 0$, but as $\|f_i(a_i)\| \le \|a_i\|$, we deduce $\|f_i(a_i)\| \longrightarrow 0$. Since the norm is non–archimedean,

$$\left\| \sum_{i=m}^{n} f_i(a_i) \right\| \le \sup_{i=m}^{n} \|f_i(a_i)\| \longrightarrow 0$$

as $m, n \to \infty$. The partial sums form a cauchy sequence, which converges in the complete metric space B.

The uniqueness of f is obvious. □

Lemma 3.4. *The category $\mathcal{S}$ is an additive category.*

Proof. Given two morphisms $f, g : A \longrightarrow B$ in $\mathcal{S}$, we form $f - g$ by the formula

$$\{f - g\}(a) = f(a) - g(a).$$

Since f and g are contractions, $\|f(a)\| \le \|a\|$ and $\|g(a)\| \le \|a\|$. This makes

$$\begin{aligned} \big\|\{f - g\}(a)\big\| &= \big\|f(a) - g(a)\big\| \\ &\le \max\big(\|f(a)\|, \|g(a)\|\big) \\ &\le \|a\|. \end{aligned}$$

Hence $f - g$ is a contraction, that is a morphism in $\mathcal{S}$.

This gives the *Hom*–sets $\mathcal{S}(A, B)$ the natural structure of abelian groups. Now observe that by Lemma 3.3 the category $\mathcal{S}$ contains countable coproducts of its objects, hence certainly finite coproducts. The reader can easily check that finite coproducts, as given in the proof of Lemma 3.3, also satisfy the universal property of finite products. Hence the category $\mathcal{S}$ is additive. □

Lemma 3.5. *The category $\mathcal{S}$ contains kernels for all its morphisms.*

Proof. Let $f : A \longrightarrow B$ be a morphism in $\mathcal{S}$. The set theoretic kernel of f, given the subspace norm in A, is a closed subgroup of A and hence complete. It is the categorical kernel. □

Lemma 3.6. *Suppose we are given countably many morphisms in $\mathcal{S}$*

$$\{f_i : A_i \longrightarrow B_i \mid 0 \le i < \infty\}.$$

The kernel of the coproduct map

$$\coprod_{i=0}^{\infty} f_i : \coprod_{i=0}^{\infty} A_i \longrightarrow \coprod_{i=0}^{\infty} B_i$$

is the coproduct of the kernels.

Proof. Both the kernel of the coproduct map and the coproduct of the kernels consist of sequences $\{a_0, a_1, a_2, \cdots\}$, with $a_i \in A_i$, so that $\|a_i\| \to 0$ and $f_i(a_i) = 0$. □

Proposition 3.7. *Let α be the cardinal $\aleph_0$. Then $\mathcal{S}$ is an additive category closed under coproducts of $\le \alpha$ of its objects, any map $s' \to s''$ may be completed to an exact sequence $s \to s' \to s''$, and coproducts of $\le \alpha$ exact sequences in $\mathcal{S}$ are exact.*

Proof. The fact that $\mathcal{S}$ is additive is Lemma 3.4. The fact that it is closed under countable coproducts is Lemma 3.3. That any map $s' \to s''$ may be completed to a short exact sequence $s \to s' \to s''$ follows from Lemma 3.5; more precisely, s may be chosen to be the kernel of $s' \to s''$. The only fact that we still have not completely proved is that coproducts of countably many exact sequences in $\mathcal{S}$ are exact.

Let $f : s' \longrightarrow s''$ be a morphism in $\mathcal{S}$, and let k be its kernel. As we said in the previous paragraph, the sequence

$$k \longrightarrow s' \longrightarrow s''$$

is exact in $\mathcal{S}$. After all, by the universal property of the kernel, the sequence

$$0 \longrightarrow \mathcal{S}(-, k) \longrightarrow \mathcal{S}(-, s') \longrightarrow \mathcal{S}(-, s'')$$

is exact in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$.

Now suppose that $s \longrightarrow s' \longrightarrow s''$ is an exact sequence in $\mathcal{S}$. Then

$$\mathcal{S}(-, s) \longrightarrow \mathcal{S}(-, s') \longrightarrow \mathcal{S}(-, s'')$$

is an exact sequence of functors. But if k is the kernel of $s' \longrightarrow s''$ as above, then we have a map $k \longrightarrow s'$ whose composite $k \longrightarrow s' \longrightarrow s''$ vanishes. By the exactness of

$$\mathcal{S}(k, s) \longrightarrow \mathcal{S}(k, s') \longrightarrow \mathcal{S}(k, s'')$$

we deduce that it factors as

$$k \xrightarrow{f} s \longrightarrow s'.$$

Also, since k is the kernel of $s' \longrightarrow s''$ and the composite $s \longrightarrow s' \longrightarrow s''$ vanishes, the map $s \longrightarrow s'$ must factor uniquely as

$$s \xrightarrow{g} k \longrightarrow s'.$$

Because the composite

$$k \xrightarrow{f} s \xrightarrow{g} k \longrightarrow s'$$

is the inclusion $k \longrightarrow s'$, it follows thar $gf : k \longrightarrow k$ is the identity. Thus the sequence

$$s \longrightarrow s' \longrightarrow s''$$

is a direct sum of the sequence

$$k \longrightarrow s' \longrightarrow s''$$

and the sequence

$$k' \longrightarrow 0 \longrightarrow 0.$$

Any exact sequence $s \longrightarrow s' \longrightarrow s''$ in $\mathcal{S}$ can be decomposed as a direct sum of a kernel, and a trivial exact sequence.

Now we need to show that a countable coproduct of exact sequences is exact. The above argument shows that it suffices to show that a countable coproduct of kernels is a kernel, and we showed that in Lemma 3.6. □

Corollary 3.8. *It follows that $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ is an abelian category (Lemma 1.3) satisfying [AB4*] (Lemma 2.1) and [AB4] (Proposition 2.9).* □

Construction 3.9. Consider now the sequence of objects and morphisms in $\mathcal{S}$

$$\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \cdots$$

where $\mathbb{Z}_p$ is the p–adic numbers with the usual norm, and the connecting maps are multiplication by p. The Yoneda functor

$$\mathcal{S} \longrightarrow \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$$

takes this to a sequence in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. We remind the reader: the Yoneda functor takes an object $s \in \mathcal{S}$ to the representable functor $\mathcal{S}(-, s)$. In the rest of this section, we will freely confuse the sequence in $\mathcal{S}$ with its image in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$.

Lemma 3.10. *The sequence*

$$\mathbb{Z}_p \xrightarrow{\ p\ } \mathbb{Z}_p \xrightarrow{\ p\ } \mathbb{Z}_p \xrightarrow{\ p\ } \cdots$$

is a sequence of monomorphisms in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$.

Proof. The kernel of $p : \mathbb{Z}_p \longrightarrow \mathbb{Z}_p$ is trivial, and hence the map

$$\mathcal{S}(-, \mathbb{Z}_p) \xrightarrow{\ p\ } \mathcal{S}(-, \mathbb{Z}_p)$$

is injective. □

Lemma 3.11. *The sequence in* $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$

$$\mathbb{Z}_p \xrightarrow{\ p\ } \mathbb{Z}_p \xrightarrow{\ p\ } \mathbb{Z}_p \xrightarrow{\ p\ } \cdots$$

has a vanishing colimit (and also a vanishing $\underrightarrow{\operatorname{colim}}^1$*).*

Proof. The colimit and $\underrightarrow{\operatorname{colim}}^1$ are, respectively, the cokernel and kernel of the map

$$\coprod_{i=0}^{\infty} \mathcal{S}(-, \mathbb{Z}_p) \xrightarrow{\ 1 - p\{shift\}\ } \coprod_{i=0}^{\infty} \mathcal{S}(-, \mathbb{Z}_p).$$

By Lemma 2.4, the natural map gives an isomorphism

$$\coprod_{i=0}^{\infty} \mathcal{S}(-, \mathbb{Z}_p) \longrightarrow \mathcal{S}\left(-, \coprod_{i=0}^{\infty} \mathbb{Z}_p\right);$$

in the commutative square below the vertical maps are isomorphisms

$$\begin{array}{ccc}
\coprod_{i=0}^{\infty} \mathcal{S}(-, \mathbb{Z}_p) & \xrightarrow{\ 1 - p\{shift\}\ } & \coprod_{i=0}^{\infty} \mathcal{S}(-, \mathbb{Z}_p) \\
\downarrow \wr & & \downarrow \wr \\
\mathcal{S}\left(-, \coprod_{i=0}^{\infty} \mathbb{Z}_p\right) & \xrightarrow{\ 1 - p\{shift\}\ } & \mathcal{S}\left(-, \coprod_{i=0}^{\infty} \mathbb{Z}_p\right)
\end{array}$$

It therefore suffices to show that the map

$$\coprod_{i=0}^{\infty} \mathbb{Z}_p \xrightarrow{\ 1 - p\{shift\}\ } \coprod_{i=0}^{\infty} \mathbb{Z}_p.$$

is an isomorphism. But its inverse is given by

$$\{1 - p\{shift\}\}^{-1} \quad = \quad 1 + p\{shift\} + p^2\{shift\}^2 + \cdots$$

and the right hand side clearly converges. □

Consider next the map of sequences in $\mathcal{S}$

$$\begin{array}{ccccccccc} A & \doteq & \mathbb{Z}_p & \xrightarrow{1} & \mathbb{Z}_p & \xrightarrow{1} & \mathbb{Z}_p & \xrightarrow{1} & \cdots \\ \downarrow & & {\scriptstyle 1}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle p^2}\downarrow & & \\ B & \doteq & \mathbb{Z}_p & \xrightarrow{p} & \mathbb{Z}_p & \xrightarrow{p} & \mathbb{Z}_p & \xrightarrow{p} & \cdots \end{array}$$

If we apply the Yoneda functor $\mathcal{S} \longrightarrow \mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$ to this map of sequences, we get a monomorphism of sequences in the abelian category $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$. We can form the quotient, deducing a short exact sequence of sequences in $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

with B the sequence of Lemma 3.11. We now prove

Proposition 3.12. *In the category $\mathcal{E}x(\mathcal{S}^{op}, \mathcal{A}b)$, there exists a sequence C of monomorphisms*

$$C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots$$

with a non–zero $\underrightarrow{\operatorname{colim}}^1$.

Proof. Let C be the sequence of monorphisms in the short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

above, where B is the sequence of Lemma 3.11. Applying the derived functor of the colimit, we have an exact sequence

$$\underrightarrow{\operatorname{colim}}^1 C \longrightarrow \underrightarrow{\operatorname{colim}} A \longrightarrow \underrightarrow{\operatorname{colim}} B.$$

By Lemma 3.11, $\underrightarrow{\operatorname{colim}} B = 0$. But A is a constant sequence

$$\mathcal{S}(-, \mathbb{Z}_p) \xrightarrow{1} \mathcal{S}(-, \mathbb{Z}_p) \xrightarrow{1} \mathcal{S}(-, \mathbb{Z}_p) \xrightarrow{1} \cdots$$

and hence $\underrightarrow{\operatorname{colim}} A = \mathcal{S}(-, \mathbb{Z}_p) \neq 0$. The exact sequence implies that $\underrightarrow{\operatorname{colim}}^1 C \neq 0$. □

Remark 3.13. The sequence B, of Lemma 3.11, is a counterexample to Proposition 5 in [4]. It is a sequence of monomorphisms, but the map $B_0 \longrightarrow \underrightarrow{\operatorname{colim}} B$ is not mono. In the absence of Proposition 5, Proposition 1 of [4] does not imply that $\underrightarrow{\operatorname{colim}}^1$ vanishes for sequences of monomorphisms. Proposition 3.12 gives an explicit example where this fails. We have an abelian category satisfying [AB4], and in it a sequence C of monomorphisms with a non-vanishing $\underrightarrow{\operatorname{colim}}^1$.

4. Appendix, by P. Deligne*: a special case

4.1. We fix a ring with unit R. *Modules* are left R-modules. By a *projective system* we will mean a projective system indexed by the ordered set $\mathbb{N}$ of integers. If $M = (M_n)_{n\in\mathbb{N}}$ is a projective system of modules, with transition maps $\varphi\colon M_{n+1} \to M_n$, the derived projective limits $\lim^0 M = \lim M$ and $\lim^1 M$ of M are the kernel and cokernel of the morphism

$$(4.1.1) \qquad 1-\varphi : \prod M_n \to \prod M_n : (x_n) \longmapsto (x_n - \varphi(x_{n+1})),$$

while the $\lim^i M$ vanish for $i \geq 2$. We define $\mathcal{A}$ to be the category of projective systems of modules for which the derived projective limit vanishes, that is, for which (4.1.1) is invertible.

Example 4.1.2. If almost all M_n are zero, then M is in $\mathcal{A}$. Indeed, the filtration of $\prod M_n$ by the $\bigoplus_{n<N} M_n$ is then a finite filtration, it is stable by $1-\varphi$, and $1-\varphi$ induces the identity on the associated graded.

Example 4.1.3. A module V with a decreasing filtration F indexed by $\mathbb{N}$ defines the projective system of the F^n. This construction is an equivalence, noted $V \mapsto F(V)$, from the category of filtered modules (V, F) with F exhaustive ($F^0 = V$) to the category of projective systems of modules with injective transition maps. If F is an exhaustive filtration of V, then $F(V)$ is in $\mathcal{A}$ if and only if the filtration F is separated and complete, that is, $V \xrightarrow{\sim} \lim V/F^n$. Indeed, we have a short exact sequence of projective systems

$$0 \to F^n \to V \to V/F^n \to 0,$$

the constant projective system V has surjective transition maps, hence a vanishing $\lim^1$, and one applies the long exact sequence of $\lim^i$:

$$0 \to \lim F^n \to V \to \lim V/F^n \to \lim^1 F^n \to 0.$$

Fix M in $\mathcal{A}$. For $m \geq n$, we denote $\varphi_{n,m}\colon M_m \to M_n$ the iterated transition map. For (x_n) in $\prod_n M_n$, the defining property of $(s_n) := (1-\varphi)^{-1}((x_n))$ is that

$$(4.1.4) \qquad s_n = \varphi(s_{n+1}) + x_n$$

If almost all x_n are zero, the sums $\sum_{m\geq n} \varphi_{n,m}(x_m)$ reduce to finite sums. They obey (4.1.4), hence

$$(4.1.5) \qquad s_n = \sum_{m\geq n} \varphi_{n,m}(x_m) \quad \text{(when almost all } x_n \text{ are zero).}$$

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In general, we define linear maps $\text{``}\sum_{m\geq n}\text{''}\varphi_{n,m}(x_m)\colon \prod M_i \to M_n$, by

$$(4.1.6)\qquad \Big(\text{``}\sum_{m\geq n}\text{''}\varphi_{n,m}(x_m)\Big)_{n\in\mathbb{N}} := (1-\varphi)^{-1}((x_n)_{n\in\mathbb{N}}).$$

As the notation suggests, $\text{``}\sum_{m\geq n}\text{''}\varphi_{n,m}(x_m)$ depends only on the x_m for $m \geq n$. This follows by linearity from the fact that if $x_i = 0$ for $i \geq n$, then s_n, given by (4.1.5), vanishes.

Proposition 4.2. *The category $\mathcal{A}$ is an abelian category in which small products and small coproducts exist and are exact, that is, $AB4$ and $AB4^*$ hold.*

Proof. Let $f\colon M \to N$ be a morphism in $\mathcal{A}$. Define K and C to be the projective systems of kernels and cokernels of the components $f_n\colon M_n \to N_n$ of f. They are the kernel and cokernel of f in the abelian category $\mathcal{P}$ of all projective systems of modules. By exactness of products in the category of modules, the products $\prod K_n$ and $\prod C_n$ are the kernel and cokernel of $(f_n)\colon \prod M_n \to \prod N_n$. The endomorphism (4.1.1) is functorial. As it is an automorphism for M and N, it is one for K and C, which hence are in $\mathcal{A}$.

The formation of (4.1.1) is compatible with products. If $(M^a)_{a\in A}$ is a family of objects of $\mathcal{A}$, the projective system of products $\Big(\prod_a M_n^a\Big)_{n\in\mathbb{N}}$ is hence again in $\mathcal{A}$: the category $\mathcal{A}$ is a full subcategory of the abelian category $\mathcal{P}$ stable by kernels, cokernels and products. It inherits from $\mathcal{P}$ being an abelian category in which small products exist and are exact. A sequence in $\mathcal{A}$ is exact if and only if it is exact in $\mathcal{P}$.

Let I be the set of the functions $f\colon A \to \mathbb{N}\cup\{\infty\}$ such that for any n in $\mathbb{N}$, the set of a in A with $f(a) \leq n$ is finite. It is suggestive to write this condition $f(a) \to \infty$ for $a \to \infty$. For each n, let $I(n)$ be the subset of I consisting of those f which are $\geq n$. When taking colimits indexed by $I(n)$, $I(n)$ will be ordered by $\geq$. As, if f and g are in $I(n)$, so is their infimum, such colimits are filtering.

Lemma 4.3. *Let $(M^a)_{a\in A}$ be a family of objects of $\mathcal{A}$, and define $M^a_\infty := 0$. The projective system S of the*

$$S_n := \operatorname{colim}_{f\in I(n)} \prod_a M^a_{f(a)}$$

is then in $\mathcal{A}$ and is the coproduct, in $\mathcal{A}$, of the M^a.

In the colimit, the transition map $\prod M^a_{f(a)} \to \prod M^a_{g(a)}$, defined for $g \leq f$, is the product of the $\varphi_{g(a),f(a)}$ (resp. 0) for $f(a) \neq \infty$ (resp. $f(a) = \infty$). The map $S_{n+1} \to S_n$ is induced by the inclusion $I(n+1) \subset I(n)$.

The components i_n^a of the natural morphism $i^a\colon\ M^a \to S$ are as follows: if $f \in I(n)$ is such that $f(a) = n$, i_n^a is the composite

$$M_n^a \to \prod_b M_{f(b)}^b \to S_n.$$

As filtering colimits and products are exact in a category of modules, Lemma 4.3 implies the existence and exactness of small coproducts in $\mathcal{A}$, and its proof will complete that of 4.2.

Example 4.3.1. Suppose that each M^a has injective transition maps, hence is deduced as in 4.1.3 from a module V^a with an exhaustive, separated and complete filtration F. We define $F^\infty(V^a) := 0$. The module S_n is then the submodule of $\prod F^n(V^a)$ consisting of the (x^a) such that for some f in $I(n)$ each x^a is in $F^{f(a)}(V^a)$. It is the completed direct sum $\lim_m \bigoplus_a F^n(V^a)/F^m(V^a)$, and the projective system S is deduced as in 4.1.3 from the completed direct sum of the V^a.

Proof of Lemma 4.3. We first prove that S is in $\mathcal{A}$. One has

$$\prod_n S_n = \prod_n \operatorname{colim}_{f \in I(n)} \prod_a M_{f(a)}^a = \operatorname{colim}_{(f_n) \in \prod I(n)} \prod_n \prod_a M_{f_n(a)}^a.$$

We map I to $\prod I(n)$ by $f \mapsto (\sup(f, n))_{n \in \mathbb{N}}$. This map is cofinal (for the order $\geq$). Indeed, for (f_n) in $\prod_n I(n)$, the infimum f of the f_n is in I, and $f_n \geq \sup(f, n)$. The colimit over the product of the $I(n)$ can hence be replaced by a colimit over I:

$$\prod S_n = \operatorname{colim}_{f \in I} \prod_n \prod_a M_{\sup(f(a),n)}^a = \operatorname{colim}_{f \in I} \prod_a \prod_n M_{\sup(f(a),n)}^a :$$

the morphism (4.1.1) for S is the colimit over f in I of the product of the morphisms (4.1.1) for the following projective systems:

$$\text{(4.3.2)} \qquad \begin{array}{ll} \text{for } f(a) \neq \infty : & M_{\sup(f(a),n)}^a \\ \text{for } f(a) = \infty : & 0 \quad . \end{array}$$

For $f(a) \neq \infty$, the projective system (4.3.2) coincides with M^a for $n \geq f(a)$. As M^a is in $\mathcal{A}$, it follows from 4.1.2 that (4.3.2) is in $\mathcal{A}$ too, and its map (4.1.1) is invertible. By passage to the colimit in f, (4.1.1) is invertible for S, and S is in $\mathcal{A}$.

It remains to check that for X in $\mathcal{A}$, the map

$$\text{(4.3.3)} \qquad u \longmapsto (ui^a)\colon\ \operatorname{Hom}(S, X) \to \prod \operatorname{Hom}(M^a, X)$$

is bijective. We first prove injectivity, i.e., that $u\colon\ S \to X$ is uniquely determined by the $u^a := ui^a\colon\ M^a \to X$.

Fix f in $I(n)$ and x in $\prod_a M^a_{f(a)}$ with image $\bar{x}$ in S_n. We have then

$$\bar{x} = \text{“}\sum_{m\geq n}\text{”}\varphi_{n,m}\Big(\sum_{f(a)=m} i^a_m(x^a)\Big),$$

where the inner sum is a finite sum in S_m, and the outer “sum” is given by (4.1.6). Indeed, for $m \geq n$, let $x^{\geq m}$ in $\prod_a M^a_{\sup(f(a),m)}$ have as components x^a, for $f(a) \geq m$, and 0 otherwise. Let $\bar{x}^{\geq m}$ be its image in S_m. We have

$$\bar{x}^{\geq m} = \varphi(\bar{x}^{\geq m+1}) + \sum_{f(a)=m} i^a_m(x^a)$$

and, as $\bar{x}^{\geq n} = \bar{x}$, the claim follows. The “sums” (4.1.6) being functorial, we get

$$u(\bar{x}) = \text{“}\sum_{m\geq n}\text{”}\varphi_{n,m}\Big(\sum_{f(a)=m} u^a_m(x^a)\Big), \tag{4.3.4}$$

computing u in terms of the u^a.

Given a family of morphisms $u_a\colon\ M^a \to X$, (4.3.4) defines a morphism $u\colon\ S \to X$ which induces the u_a. To prove the surjectivity of (4.3.3), one has to check that this definition of u is legitimate, i.e., that for $f \geq g$ in $I(n)$, the diagram

$$\begin{array}{ccc} \prod M^a_{f(a)} & \longrightarrow & \prod M^a_{g(a)} \\ & {\scriptstyle (4.3.4)}\searrow \quad \swarrow{\scriptstyle (4.3.4)} & \\ & X_n & \end{array}$$

is commutative. Fix (x^a) in $\prod M^a_{f(a)}$. For $p \geq q$, let y_{pq} in X_p be the sum of the $u^a_p(x^a)$ for $f(a) = p$ and $g(a) = q$. For each fixed q, the set of a for which $g(a) = q$ is finite, hence only finitely many y_{pq} are not zero. We have to check that

$$\text{“}\sum_{p\geq n}\text{”}\varphi_{n,p}\Big(\sum_q y_{p,q}\Big) = \text{“}\sum_{q\geq n}\text{”}\varphi_{n,q}\Big(\sum_p \varphi_{q,p}(y_{p,q})\Big).$$

The difference is “$\sum_{p\geq n}$”$\varphi_{n,p}$ applied to the sum in $\prod X_n$

$$\sum_{p,q}\Big((y_{p,q} \text{ in } X_p) - (\varphi_{q,p}(y_{p,q}) \text{ in } X_q)\Big)$$
$$= \sum_{p,q}\Big(\sum_{p\geq i>q} (1-\varphi)(\varphi_{i,p}(y_{p,q}) \text{ in } X_i)\Big)$$

This sum is $(1-\varphi)$ applied to an element of $\prod_{p>n} X_p$, and its “$\sum_{p\geq n}$”$\varphi_{n,p}$ hence vanishes.

4.4. For a an integer, define the filtered module $R(a)$ to be R, purely in filtration a. By 4.1.2 or 4.1.3, the corresponding projective system $T^a := F(R(a))$ is in $\mathcal{A}$. It is given by $T^a_n = R$ for $n \leq a$, 0 otherwise. The morphism $R(a) \to R(a+1)$ which is the identity on the underlying modules induces a monomorphism of projective systems $T^a \to T^{a+1}$:

$$\begin{array}{ccccccccccccccc} T^a & : \cdots \to & 0 & \to \cdots \to & 0 & \to & 0 & \to & R & \to \cdots \to & R \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ T^{a+1} & : \cdots \to & 0 & \to \cdots \to & 0 & \to & R & \to & R & \to \cdots \to & R \end{array}$$

.

Proposition 4.4.1. *In the category* $\mathcal{A}$*,* $\operatorname{colim} T^a = 0$.

Proof. One has to show that for any X in $\mathcal{A}$, $\lim_a \operatorname{Hom}(T^a, X) = 0$. Indeed,

$$\operatorname{Hom}(T^a, X) = X_a, \tag{4.4.2}$$

and $\lim\ X_a = 0$.

Remark 4.4.3. As in 3.12, one can deduce from 4.4.1 the existence of inductive sequences with injective transition maps in $\mathcal{A}$, for which the left derived colim^1 does not vanish. Example: the inductive system of the $\operatorname{coker}(T^0 \to T^a)$. In the opposite category, this becomes a projective system with surjective transition maps for which $\lim^1$ does not vanish.

4.5. We now assume that R is a field. In 1.2, we take $\alpha = \aleph_0$. Let $\mathcal{S}$ be the category of filtered modules (V, F) as in 4.1.3, with F exhaustive, separated and complete, and in addition such that each $\operatorname{Gr}^n_F(V)$ is of countable dimension. Countable coproducts exist in $\mathcal{S}$: the coproduct in $\mathcal{S}$ of a countable family of objects (V_i, F) of $\mathcal{S}$ is the completed direct sum

$$\coprod_i V_i = \lim\Big(\bigoplus_i V_i/F^n\Big).$$

We found the category $\mathcal{A}$ by unravelling the definition of $\mathcal{E}x(\mathcal{S}^{\mathrm{op}}, Ab)$. With the notation $R(n)$ of 4.4, if T is an additive contravariant functor from $\mathcal{S}$ to Ab, each $T(R(n))$ has a natural structure of R-vector space. Indeed, in $\mathcal{S}$, $R(n)$ has a right R-module structure. We define $M(T)$ to be the projective system of the vector spaces $T(R(n))$.

Proposition 4.5.1. *With the notation above, the functor* $T \mapsto M(T)$ *induces an equivalence of categories from* $\mathcal{E}x(\mathcal{S}^{\mathrm{op}}, Ab)$ *to* $\mathcal{A}$

Proof. We first prove that when T is in $\mathcal{E}x(\mathcal{S}^{\mathrm{op}}, Ab)$, $M(T)$ is in $\mathcal{A}$. Indeed,

$$\prod M_n(T) = \prod T(R(n)) = T\Big(\coprod R(n)\Big),$$

the maps $R(n) \to R[t]$, $x \mapsto xt^n$, identify the ordinary direct sum of the $R(n)$ with $R[t]$, provided with the t-adic filtration, and the completed direct sum $\coprod R(n)$ is $R[[t]]$, with the t-adic filtration. The endomorphism (4.1.1) for $M(T)$ is the image by T of the endomorphism $1 - t$ of $R[[t]]$. As the latter is invertible, $M(T)$ is in $\mathcal{A}$.

Let $\mathcal{S}_0$ be the full subcategory of $\mathcal{S}$ consisting of the finite sums of $R(n)$ (the (V, F) in $\mathcal{S}$ with V finite dimensional). Evaluation at the $R(n)$ is an equivalence from the category $\mathcal{H}om_{\text{add}}(\mathcal{S}_0^{\text{op}}, Ab)$ of contravariant additive functors from $\mathcal{S}_0$ to Ab, to the category $\mathcal{P}$ of projective systems. The functor $T \mapsto M(T)$, from $\mathcal{E}x(\mathcal{S}^{\text{op}}, Ab)$ to $\mathcal{A}$, becomes in this way the functor $T \mapsto T^0$ of restriction to $\mathcal{S}_0$.

The construction 4.1.3 turns $\mathcal{S}$ into a full subcategory of $\mathcal{A}$, and, by 4.3.1, countable direct sums in $\mathcal{S}$ are direct sums in $\mathcal{A}$. If to each A in $\mathcal{A}$ we attach the restriction to $\mathcal{S}$ of the representable functor h_A, we hence obtain

$$\mathcal{A} \to \mathcal{E}x(\mathcal{S}^{\text{op}}, Ab).$$

By (4.4.2), the composite $\mathcal{A} \to \mathcal{E}x(\mathcal{S}^{\text{op}}, Ab) \to \mathcal{A}$ is isomorphic to the identity. It remains to check that the composite $\mathcal{E}x(\mathcal{S}^{\text{op}}, Ab) \to \mathcal{A} \to \mathcal{E}x(\mathcal{S}^{\text{op}}, Ab)$ is isomorphic to the identity as well. This composite sends a functor T to the functor

$$V \longmapsto \text{Hom}\big(h_V^0, T^0\big)$$

(Hom in $\mathcal{H}om_{\text{add}}(\mathcal{S}_0, Ab)$). By Yoneda's lemma, the functor T can be identified with $V \longmapsto \text{Hom}(h_V, T)$. Restriction to $\mathcal{S}_0$ defines

$$T(V) = \text{Hom}(h_V, T) \to \text{Hom}\big(h_V^0, T^0\big). \tag{4.5.2}$$

and it remains to check that for V in $\mathcal{S}$ and T in $\mathcal{E}x(\mathcal{S}^{\text{op}}, Ab)$, (4.5.2) is an isomorphism. In other words, a system of $t_{X,e} \in T(X)$, functorial in $X, e\colon X$ in $\mathcal{S}_0$ and $e\colon X \to V$, should come from a unique $t \in T(V)$.

For each n, let us lift in $F^n(V)$ a basis $(e_{n,\alpha})_{\alpha \in A_n}$ of F^n/F^{n+1}. This gives us morphisms $e(n,\alpha)\colon R(n)_\alpha \to V$, where $R(n)_\alpha$ is a copy of $R(n)$. The resulting morphism

$$\coprod_{n,\alpha} R(n)_\alpha \to V$$

countable coproduct in $\mathcal{S}$, indexed by $A = \coprod A_n$, of 1-dimensional objects. By assumption, $T(V) \xrightarrow{\sim} \prod_{n,\alpha} T(R(n)_\alpha)$. The injectivity of this map proves that of (4.5.2).

If $(t_{X,e})$ is in $\text{Hom}(h_V^0, T^0)$, the $t_{R(n)_\alpha, e(n,\alpha)}$ come from a unique $t \in T(V)$. To check surjectivity of (4.5.2), it remains to see that if the $t_{R(n)_\alpha, e(n,\alpha)}$ are zero, so are all $t_{X,e}$, and it suffices to check this when X is a $R(k)$. The map $e\colon R(k) \to V$ is a sum $\sum_{n \geq k} f_n$, where f_n factors through a finite sum of $R(n)_\alpha$. For $\ell \geq k$, the morphism $\sum_{n \geq \ell} f_n$ factors through $e_\ell\colon R(\ell) \to V$. The difference of $e_\ell\colon R(\ell) \to V$ and of $R(\ell) \xrightarrow{\varphi} R(\ell+1) \xrightarrow{e_{\ell+1}} V$ factors through a finite sum of $R_{n,\alpha}$, hence $t_{R(\ell), e_\ell - e_{\ell+1} \circ \varphi} = 0$ and $\varphi(t_{R(\ell+1), e_{\ell+1}}) = t_{R(e), e_\ell}$. As $\lim T(R(\ell)) = 0$, it follows that the $t_{R(k), e_\ell}$ and in particular $t_{R(k), e}$ are zero.

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