Arithmetic hyperbolic manifolds

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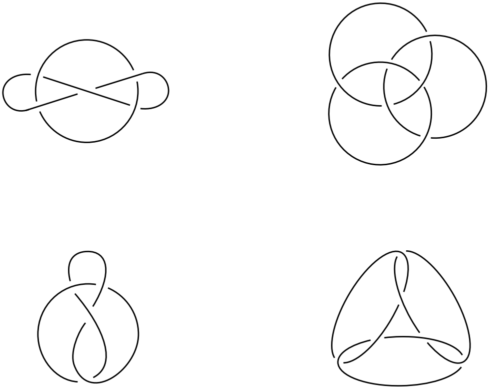
Many of the key examples in the development of the theory of geometric structures on 3-manifolds (e.g. the figure-eight knot complement, the Whitehead link complement, the complement of the Borromean rings and the Magic manifold) are arithmetic.



The modular group

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Every non-cocompact finite co-area arithmetic Fuchsian group is commensurable with the modular group.

Some particularly interesting subgroups of $\mathrm{PSL}(2, \mathbf{Z})$ of finite index are the congruence subgroups.

A subgroup $\Gamma < \mathrm{PSL}(2, \mathbf{Z})$ is called a congruence subgroup if there exists an $n \in \mathbf{Z}$ so that Γ contains the principal congruence group:

$$\Gamma(n) = \ker\{\mathrm{PSL}(2, \mathbf{Z}) \to \mathrm{PSL}(2, \mathbf{Z}/\mathrm{n}\mathbf{Z})\},$$

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n is called the level.

The structure of congruence subgroups (genus, torsion, number of cusps) has been well-studied.

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This was proved by J. B. Denin in the 70's.

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Note $\mathrm{PSL}(2, \mathbf{Z})/\mathrm{N} \cong$ the $(2, 3, n)$ triangle group.

3. Claim: $\Gamma(n)$ is generated by parabolic elements if and only if $N = \Gamma(n)$.

Given the claim the result follows as if $N = \Gamma(n)$ then N has finite index; i.e. the the $(2, 3, n)$ triangle group is finite.

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Now $\mathbf{H}^2/\mathrm{PSL}(2, \mathbf{Z})$ has 1 cusp. So if $\Gamma(n)$ is generated by parabolic elements $\{p_1, \ldots p_r\}$, then each p_i is $\mathrm{PSL}(2, \mathbf{Z})$-conjugate into $\left\langle \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \right\rangle$; i.e., $p_i \in N$.

A more general version of Rademacher's Conjecture that we now discuss was proved by J. G. Thompson and independently P. Zograf.

Suppose $\Gamma < \mathrm{PSL}(2, \mathbf{R})$ is commensurable with $\mathrm{PSL}(2, \mathbf{Z})$. Define Γ to be a congruence subgroup if Γ contains some $\Gamma(n)$.

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Examples: Suppose $n > 1$ and let $\Gamma_0(n) < \mathrm{PSL}(2,\mathbf{Z})$ denote the subgroup consisting of those elements congruent to $\pm \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ $\pmod{n}$.

Note that $\tau_n = \begin{pmatrix} 0 & -1/\sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix}$ normalizes $\Gamma_0(n)$.

Hence $\langle \Gamma_0(n), \tau_n \rangle \subset N_{\mathrm{PSL}(2,\mathbf{R})}(\Gamma_0(n))$ is

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Remarks: 1. The groups $N_{\mathrm{PSL}(2,\mathbf{R})}(\Gamma_0(n))$ contain all maximal Fuchsian groups commensurable with $\mathrm{PSL}(2,\mathbf{Z})$.

2. These involutions illustrate a common theme in arithmetic groups—lots of hidden symmetry!

These involutions are hidden to $\mathrm{PSL}(2,\mathbf{Z})$ but visible on finite index subgroups.

Theorem 1 (Thompson, Zograf)

There are only finitely many congruence Fuchsian groups commensurable with $\mathrm{PSL}(2, \mathbf{Z})$ *of genus 0 (resp. of fixed genus).*

Sketch Proof: By Selberg's work congruence groups have a spectral gap; i.e. if Γ is congruence then $\lambda_1(\mathbf{H}^2/\Gamma) \geq 3/16$.

On the other hand we have the following result of Zograf:

Theorem 2

Let Γ *be a Fuchsian group of finite co-area and let the genus of* $\mathbf{H}^2/\Gamma$ *be denoted by* $g(\Gamma)$. *If* $Area(\mathbf{H}^2/\Gamma) \geq 32\pi(g(\Gamma)+1)$, *then*

$$\lambda_1(\Gamma) < \frac{8\pi(g(\Gamma)+1)}{Area(\mathbf{H}^2/\Gamma)}.$$

Now let Γ_i be a sequence of congruence subgroups of genus 0.

There are only finitely many arithmetic Fuchsian groups of bounded co-area.

Thus areas $\to \infty$ and so by Zograf:

$$\lambda_1 \to 0.$$

This is a contradiction, since by Selberg there is a spectral gap for congruence subgroups.

Maximal Groups of genus 0:

Below we list those n for which the maximal groups (as constructed above) have genus 0. The case of $n = 1$ is the modular group.

Prime Level: 2,3,5,7,11,13,17,19,23,29,31,41,47,59,71.

Non-prime Level: 6,10, 14, 15,21,22,26,30, 33,34,35,38,39,42,51,55,62,66,69, 70,78,87, 94,95,105,110, 119,141.

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Non-prime Level: 6,10, 14, 15,21,22,26,30, 33,34,35,38,39,42,51,55,62,66,69, 70,78,87, 94,95,105,110, 119,141.

As Ogg noticed back in the 70's:

the prime values are precisely the prime divisors of the order of the Monster simple group.

There are 132 genus 0 congruence subgroups of $\mathrm{PSL}(2, \mathbf{Z})$ (up to conjugacy in $\mathrm{PSL}(2, \mathbf{Z})$ (C. K. Seng, M. L. Lang, Y. Yifan, 2004)

There are 132 genus 0 congruence subgroups of $\mathrm{PSL}(2, \mathbf{Z})$ (up to conjugacy in $\mathrm{PSL}(2, \mathbf{Z})$) (C. K. Seng, M. L. Lang, Y. Yifan, 2004)

26 of these are torsion free (A. Sebbar, 2001)

TABLE 1.

Index	Level	Group
6	2	$\Gamma(2)$
	4	$\Gamma_0(4)$
12	3	$\Gamma(3)$
	4	$\Gamma_0(4)\cap\Gamma(2)$
	5	$\Gamma_1(5)$
	6	$\Gamma_0(6)$
	8	$\Gamma_0(8)$
	9	$\Gamma_0(9)$
24	4	$\Gamma(4)$
	6	$\Gamma_0(3)\cap\Gamma(2)$
	7	$\Gamma_1(7)$
	8	$\Gamma_1(8),\ \Gamma_0(8)\cap\Gamma(2),\ \left\{\pm\begin{pmatrix}1+4a & 2b\\ 4c & 1+4d\end{pmatrix},\ a\equiv c \mod 2\right\}$
	12	$\Gamma_0(12)$
	16	$\Gamma_0(16),\ \left\{\pm\begin{pmatrix}1+4a & b\\ 8c & 1+4d\end{pmatrix},\ a\equiv c \mod 2\right\}$
36	6	$\Gamma_0(2)\cap\Gamma(3)$
	9	$\Gamma_1(9),\ \left\{\pm\begin{pmatrix}1+3a & 3b\\ 3c & 1+3d\end{pmatrix},\ a\equiv c \mod 3\right\}$
	10	$\Gamma_1(10)$
	18	$\Gamma_0(18)$
	27	$\left\{\pm\begin{pmatrix}1+3a & b\\ 9c & 1+3d\end{pmatrix},\ a\equiv c \mod 3\right\}$
48	8	$\Gamma_1(8)\cap\Gamma(2),\ \left\{\pm\begin{pmatrix}1+4a & 4b\\ 4c & 1+4d\end{pmatrix},\ a\equiv c \mod 2\right\}$
	12	$\Gamma_1(12),\ \left\{\pm\begin{pmatrix}1+6a & 2b\\ 6c & 1+6d\end{pmatrix},\ a\equiv c \mod 2\right\}$
	16	$\Gamma_0(16)\cap\Gamma_1(8),\ \left\{\pm\begin{pmatrix}1+4a & 2b\\ 8c & 1+4d\end{pmatrix},\ a\equiv c \mod 2\right\}$
	24	$\left\{\pm\begin{pmatrix}1+6a & b\\ 12c & 1+6d\end{pmatrix},\ a\equiv c \mod 2\right\}$
	32	$\left\{\pm\begin{pmatrix}1+4a & b\\ 16c & 1+4d\end{pmatrix},\ a\equiv c \mod 2\right\}$
60	5	$\Gamma(5)$
	25	$\Gamma_0(25)\cap\Gamma_1(5)$

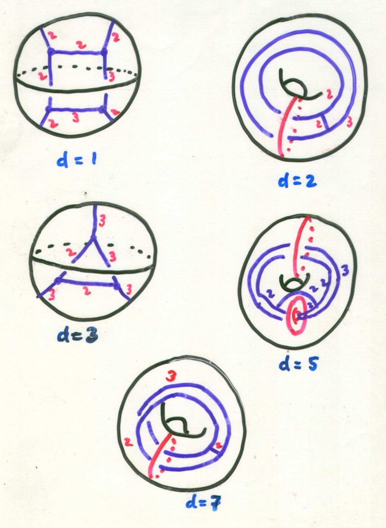
Abdellah Sebbar, Proc. AMS 129 (2001)

Dimension 3: The Bianchi groups

Let d be a square-free positive integer, and O_d the ring of integers of the quadratic imaginary number field $\mathbf{Q}(\sqrt{-d})$.

The Bianchi groups are defined to be the family of groups $\mathrm{PSL}(2, \mathrm{O_d})$. Let $Q_d = \mathbf{H}^3/\mathrm{PSL}(2, \mathrm{O_d})$ denote the Bianchi orbifold.

Some Bianchi Orbifolds(from Hatcher's paper)



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Natural generalization of genus 0 surface groups are link groups in S^3.

The Cuspidal Cohomology Problem

Theorem 3 (Vogtmann finishing off work by lots of Germans)

If $S^3 \setminus L \to Q_d$ *then* $d \in \{1, 2, 3, 5, 6, 7, 11, 15, 19, 23, 31, 39, 47, 71\}$.

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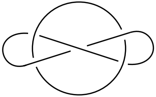
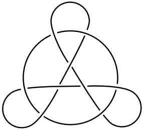
Theorem 4 (M. Baker)

For each d *in this list there is a link* L_d *such that* $S^3 \setminus L_d \to Q_d$.

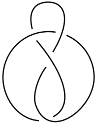
Saw some examples earlier for some small values of d. Here are some more:



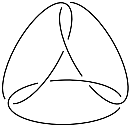
$d = 1$



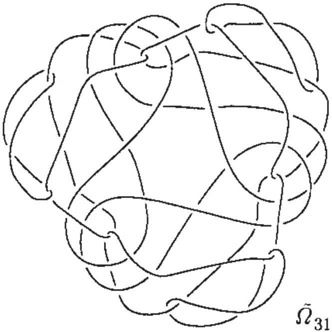
$d = 2$



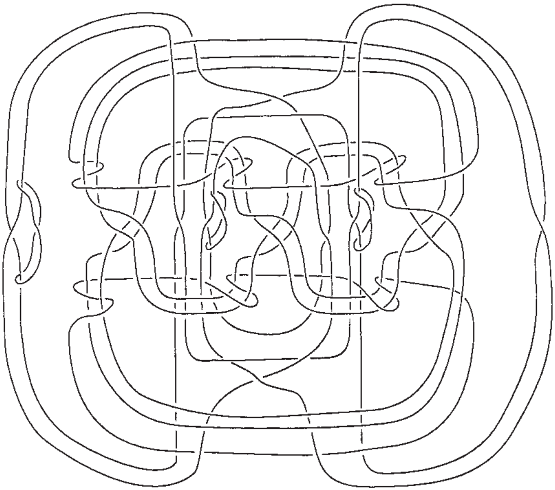
$d = 3$



$d = 7$



$\tilde{\Omega}_{31}$



$\tilde{\Omega}_{47}$

Call a link $L \subset S^3$ *arithmetic* if $S^3 \setminus L = \mathbf{H}^3/\Gamma$ where Γ is arithmetic (in this case we mean commensurable with $\mathrm{PSL}(2, \mathrm{O_d})$)

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Remarks:1. The figure eight knot is the only arithmetic knot.

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Remarks:1. The figure eight knot is the only arithmetic knot.

2. There are infinitely many arithmetic links–even with two components.

Congruence link complements

We define congruence subgroups, principal congruence subgroups as above; ie

A subgroup $\Gamma < \mathrm{PSL}(2, \mathrm{O_d})$ is called a congruence subgroup if there exists an $I \subset O_d$ (as before called the level) so that Γ contains the principal congruence group:

$$\Gamma(I) = \ker\{\mathrm{PSL}(2, \mathrm{O_d}) \to \mathrm{PSL}(2, \mathrm{O_d}/\mathrm{I})\},$$

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Question 1 (Analogue of Rademacher's Conjecture)

Are there only finitely many congruence link complements in S^3?

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Question 2

Is there some version of Ogg's observation—i.e. which maximal groups have trivial cuspidal cohomology? Infinitely many prime levels?

Lemma 5

There are only finitely many principal congruence link complements in S^3.

Proof.

Note by Vogtmann's result, only finitely many possible d's.

If $M = \mathbf{H}^3/\Gamma(I)$ is a link complement in S^3, then some cusp torus contains a short curve (length < 6). The peripheral subgroups have entries in I. As the norm of the ideal I grows then elements in I have absolute values > 6. □

An alternative approach is using Systole bounds.

Theorem 6 (Adams-R, 2000)

Let N be a closed orientable 3-manifold which does not admit any Riemannian metric of negative curvature. Let L be a link in N whose complement admits a complete hyperbolic structure of finite volume. Then $sys(N \setminus L) \leq 7.35534\ldots$.

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For principal congruence manifolds the following simple lemma shows that systole will grow with the norm of the ideal.

Lemma 7

Let $\gamma \in \Gamma(I)$ be a hyperbolic element. Then $\mathrm{tr}\ \gamma = \pm 2 \mod I^2$.

Using this idea G. Lakeland and C. Leininger recently proved:

Theorem 8 (Lakeland-Leininiger)

Let M be a closed orientable 3-manifold, then there are only finitely many principal congruence subgroups with $M \setminus L \cong \mathbf{H}^3/\Gamma(I)$.

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Question 3 (Generalized Rademacher's Conjecture)

For M a fixed closed orientable 3-manifold are there only finitely many congruence link complements in M?

Thurston in an email in 2009: “Although there are infinitely many arithmetic link complements, there are only finitely many that come from principal congruence subgroups. Some of the examples known seem to be among the most general (given their volume) for producing lots of exceptional manifolds by Dehn filling, so I’m curious about the complete list.”

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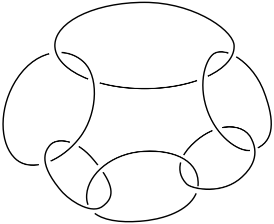
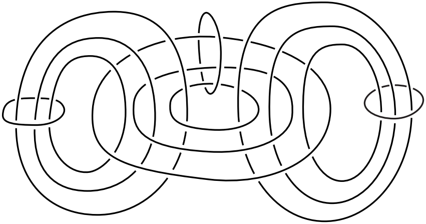
What are the principal congruence link complements?

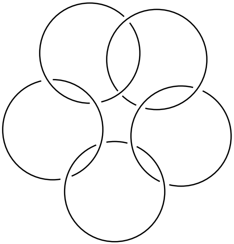
Old Examples (from Baker's thesis): All levels are 2



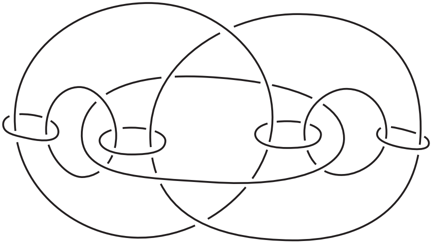
d=1



d=2



d=3

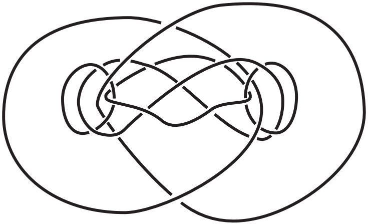


d=7

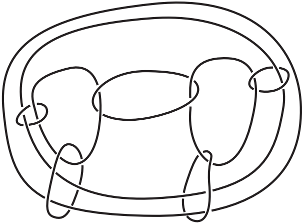
The Magic manifold is principal congruence of level $\langle(1+\sqrt{-7})/2\rangle$. Generators for the fundamental group are (from Grunewald-Schwermer):

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & (1+\sqrt{-7})/2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -(1+\sqrt{-7})/2 & 1 \end{pmatrix}.$$

Some other examples in $d = 15$ and $d = 23$ of levels $\langle 2, (1+\sqrt{-15})/2 \rangle$ and $\langle 2, (1+\sqrt{-23})/2 \rangle$



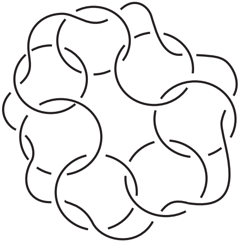
d=15



d=23

Thurston's principal congruence link complement

For $d = 3$ and level $\langle (5+\sqrt{-3})/2 \rangle$, Thurston observed that the complement of the link below is a principal congruence link complement.



Thurston's principal congruence link

He said in an email in 2009:“*One of the most intriguing congruence covers I know is for the ideal generated by* $(5+\sqrt{-3})/2$ *in* $\mathrm{PSL}(2,\mathbf{Z}[\omega])$ *which is an 8-component link complement in* S^3.”

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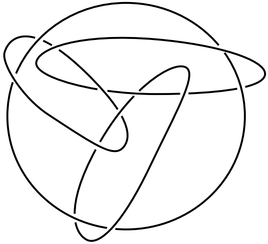
Indeed even in his notes there are examples. Doesn’t say it’s principal congruence but he probably knew!

$L_1 =$



$d = 2$, level $=< 1+\sqrt{-2} >$

Another nice example is from Hatcher's JLMS paper. Here $d = 11$ and the ideal is $\langle(1+\sqrt{-11})/2)\rangle$. Need to prove it is principal congruence.

$L_2 =$

Hatcher's Example

The following theorem contains all known principal congruence link complements. This contains “old” examples, and “new” ones. This includes examples from M. Baker and myself and also work of Matthias Goerner (2011 Berkeley thesis) and his recent preprint (arXiv:1406.2827) Regular Tessellation Links.

Theorem 9

The following list of pairs (d, I) *indicates the known Bianchi groups* $\mathrm{PSL}(2, \mathrm{O_d})$ *containing a principal congruence subgroup* $\Gamma(I)$ *such that* $\mathbf{H}^3/\Gamma(I)$ *is a link complement in* S^3*. Those annotated by * are new.*

$d = 1$: $I \in \{2, \langle 2 \pm i\rangle^*, \langle (1 \pm i)^3\rangle^*, 3^*, \langle 3 \pm i\rangle^*, \langle 3 \pm 2i\rangle^*, \langle 4 \pm i\rangle^*\}$.

$d = 2$: $I \in \{2, \langle 1 \pm \sqrt{-2}\rangle^*, \langle 2 \pm \sqrt{-2}\rangle^*\}$.

$d = 3$: $I \in \{2, 3, \langle (5 \pm \sqrt{-3})/2\rangle, \langle 3 \pm \sqrt{-3}\rangle, \langle (7 \pm \sqrt{-3})/2\rangle^*, \langle 4 \pm \sqrt{-3}\rangle^*, \langle (9 \pm \sqrt{-3})/2\rangle^*\}$.

$d = 5$: $I = \langle 3, 1 \pm \sqrt{-5}\rangle^*$.

$d = 7$: $I \in \{\langle (1 \pm \sqrt{-7})/2\rangle, 2, \langle (3 \pm \sqrt{-7})/2\rangle^*, \langle 1 \pm \sqrt{-7}\rangle^*\}$.

$d = 11$**:** $I \in \{\langle(1 \pm \sqrt{-11})/2\rangle^*, \langle(3 \pm \sqrt{-11})/2\rangle^*\}$.

$d = 15$**:** $I = \langle 2, (1 \pm \sqrt{-15})/2\rangle$.

$d = 19$**:** $I = \langle(1 \pm \sqrt{-19})/2\rangle$.

$d = 23$**:** $I = \langle 2, (1 \pm \sqrt{-23})/2\rangle$.

$d = 31$**:** $I = \langle 2, (1 \pm \sqrt{-31})/2\rangle$.

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Goerner also shows this is a complete list in the cases of $d = 1, 3$.

$d = 11$: $I \in \{\langle (1 \pm \sqrt{-11})/2 \rangle^*, \langle (3 \pm \sqrt{-11})/2 \rangle^* \}$.

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This leaves $d = 6, 39, 47, 71$.

Recent work with Baker suggest none when $d = 6$.

In the case when the level is a rational integer we can say more.

Theorem 10 (Baker-R)

Let $n \in \mathbf{Z}$. Then $\Gamma(n) < \mathrm{PSL}(2, \mathrm{O_d})$ is a link group in S^3 if and only if:

$$(d, n) \in \{(1,2),(2,2),(3,2),(7,2),(1,3),(3,3)\}.$$

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Let $L = L_1 \cup \ldots \cup L_n \subset S^3$ be a link, $X(L)$ denote the exterior of L, and $\Gamma = \pi_1(S^3 \setminus L)$ be the link group. Then:

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1. Γ^{ab} *is torsion-free of rank equal to the number of components of* L*; i.e.* $\Gamma^{ab} \cong \mathbf{Z}^n$.
2. Γ *is generated by parabolic elements.*
3. *For each component* L_i*, there is a curve* $x_i \subset \partial X(L)$ *so that Dehn filling* $S^3 \setminus L$ *along the totality of these curves gives* S^3*. Following Perelman's resolutio n of the Geometrization Conjecture, this can be rephrased as saying that the group obtained by setting* $x_i = 1$ *for each* i *is the trivial group.*

Given this, our method is:

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Note that the converse also holds in the case when Q_d has 1 cusp. For if $\Gamma(I)$ is generated by parabolic elements, then since $\Gamma(I)$ is a normal subgroup and Q_d has 1 cusp, all such generators are $\mathrm{PSL}(2, \mathrm{O_d})$-conjugate into P.

The orders of the groups $\mathrm{PSL}(2, \mathrm{O_d}/\mathrm{I})$ are known, and we can use Magma to test whether $\Gamma(I) = \langle P \rangle$.

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Sometimes this does not work!

Step 2: *Find parabolic elements in* $\Gamma(I)$ *so that as above, trivializing these elements, trivializes the group.*

This step is largely done by trial and error, however, the motivation for the idea is that, if $\mathbf{H}^3/\Gamma$ has n cusps, we attempt to find n parabolic fixed points that are $\Gamma(I)$-inequivalent, and for which the corresponding parabolic elements of $\langle P\rangle$ provide curves that can be Dehn filled above.

Example The case of $d = 1$. $\Gamma(\langle 2+i\rangle)$ is a six component link group.

$\mathrm{PSL}(2, \mathrm{O}_1) = \langle \mathrm{a}, \ell, \mathrm{t}, \mathrm{u} \mid \ell^2 = (\mathrm{t}\ell)^2 = (\mathrm{u}\ell)^2 = (\mathrm{a}\ell)^2 = \mathrm{a}^2 = (\mathrm{ta})^3 = (\mathrm{ua}\ell)^3 = 1, [\mathrm{t}, \mathrm{u}] = 1\rangle$.

(i) $N(\langle 2+i\rangle) = 5$, so $\Gamma(\langle 2+i\rangle)$ is a normal subgroup of $\mathrm{PSL}(2, \mathrm{O}_1)$ of index 60.

(ii) The image of the peripheral subgroup in $\mathrm{PSL}(2, \mathrm{O}_1)$ fixing ∞ under the reduction homomorphism is dihedral of order 10. Hence $\mathbf{H}^3/\Gamma(\langle 2+i\rangle)$ has 6 cusps.

(iii) Use Magma as discussed above to see that $[\mathrm{PSL}(2, \mathrm{O}_1) : \langle \mathrm{P}\rangle] = 60$, and so $\Gamma(\langle 2+i\rangle) = \langle P\rangle$.

Finding inequivalent cusps and the right parabolics

Lemma 11

Let $S = \{\infty, 0, \pm 1, \pm 2\}$. Then each element of S is a fixed point of some parabolic element of $\Gamma(\langle 2+i\rangle)$ and moreover they are all mutually inequivalent under the action of $\Gamma(\langle 2+i\rangle)$. The parabolics are:

$$S' = \{t^2u, at^2ua, t^{-1}at^2uat, tat^2uat^{-1}, t^{-2}at^2uat^2, t^2at^{-3}uat^{-2}\}.$$

Magma routine

$G\langle a,l,t,u\rangle := Group\langle a,l,t,u|l^2,a^2,(t*l)^2,(u*l)^2,$

$(a*l)^2,(t*a)^3,(u*a*l)^3,(t,u)\rangle;$

$h := sub\langle G|t^2*u,t^5\rangle;$

$n := NormalClosure(G,h);$

$print\ Index(G,n);$

60

$print\ AbelianQuotientInvariants(n);$

[0, 0, 0, 0, 0, 0]

$r := sub\langle n|t^2*u,a*t^2*u*a,t^{-1}*a*t^2*u*a*t,t*a*t^2*u*a*$

$t^{-1},t^{-2}*a*t^2*u*a*t^2,t^2*a*t^{-3}*u*a*t^{-2}\rangle;$

$print\ Index(n,r);$

1

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As in the case of dimension 2, Congruence manifolds have a spectral gap: here $\lambda_1 \geq 3/4$ (should be 1).

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As in the case of dimension 2, Congruence manifolds have a spectral gap: here $\lambda_1 \geq 3/4$ (should be 1).

However:

Theorem 12 (Lackenby-Souto)

There exists an infinite family of links $\{L_n\}$ *in* S^3 *with* $\mathrm{Vol}(\mathrm{S}^3 \setminus \mathrm{L_n}) \to \infty$ *such that* $\lambda_1 > C > 0$ *for some constant* C.

Problem Can't use spectral gap directly!

Corollary: No Zograf theorem in dimension 3.

Call a family of links as in the Lackenby-Souto theorem, an expander family.

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Corollary 14

There are finitely many congruence alternating link complements.

There is also other work by Futer-Kalfagianni-Purcell constructing other non expander families.

Amusing side note:

In an email correspondence with Thurston about congruence links I mentioned Lackenby's result and he said the following :

"I wasn't familiar with Lackenby's work, but alternating knots are related in spirit to Riemannian metrics on S^2, which does not admit an expander sequence of metrics, so alternating links are not the best candidates."

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"I wasn't familiar with Lackenby's work, but alternating knots are related in spirit to Riemannian metrics on S^2, which does not admit an expander sequence of metrics, so alternating links are not the best candidates."

He then proceed to outline a construction to produce an expander family of links.

What can one use?

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Expectation Sequences of congruence subgroups should develop torsion in H_1.

This would rule out infinitely many congruence link complements.

Where are the manifolds with large λ_1?

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Theorem 15 (Long-Lubotkzy-R)

Let Γ be a finite co-volume Kleinian group. Then Γ contains a nested descending tower of normal subgroups

$$\Gamma > N_1 > N_2 > \ldots > N_k > \ldots \text{ with } \cap N_i = 1$$

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and $C > 0$ with $\lambda_1(N_i) > C$.

Question 4

Can these ever be link groups in S^3?

Back to dimension 2

Back to dimension 2

There is a notion of congruence subgroup for cocompact Fuchsian groups(or lattices more generally). As before such examples have a spectral gap.

Question 5

Are there congruence surfaces of every genus?

Remark: There are only finitely many conjugacy class of arithmetic surface groups of genus g.

Expectation: (i) These congruence surfaces will not lie in a fixed commensurability class.

(ii) These congruence surfaces will not arise from invariant trace-fields of bounded degree.

Remarks: 1. Brooks and Makover gave a construction of closed surfaces of every genus (non-arithmetic) with "large" λ_1.

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Question 6

Can one build surfaces of every genus in a fixed commensurability class with a spectral gap?

Remark: Kassabov proved that S_n and A_n can be made expanders for certain choices of generators. Can S_n and A_n be made expanders on 2 generators?

If so then can build surfaces as in the previous question.

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THE END