Introduction to Representations Theory of Lie Groups

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Introduction

The purpose of this notes is twofold. The first goal is to give a quick answer to the question "What is representation theory about?" To answer this, we will show by examples what are the most important results of this theory, and the problems that it is trying to solve. To make the answer short we will not develop all the formal details of the theory and we will give preference to examples over proofs. Few results will be proved, and in the ones were a proof is given, we will skip the technical details. We hope that the examples and arguments presented here will be enough to give the reader and intuitive but concise idea of the covered material.

The second goal of the notes is to be a guide to the reader interested in starting a serious study of representation theory. Sometimes, when starting the study of a new subject, it's hard to understand the underlying motivation of all the abstract definitions and technical lemmas. It's also hard to know what is the ultimate goal of the subject and to identify the important results in the sea of technical lemmas. We hope that after reading this notes, the interested reader could start a serious study of representation theory with a clear idea of its goals and philosophy.

Lets talk now about the material covered on this notes. In the first section we will state the celebrated Peter-Weyl theorem, which can be considered as a generalization of the theory of Fourier analysis on the circle S^1. Lets recall that Fourier theory says that the functions $\{f_n(x) = e^{ln\,x}\}$ form a Hilbert basis for the space $L^2(S^1)$. The relation between this result and representation theory is the following: Let G be a compact group, and let (π, V) be an irreducible representation of G, i.e., a morphism

$$\pi : G \longrightarrow GL(V)$$

such that the only invariant subspaces of V are $\{0\}$ and V. Given a basis $\{v_1, \dots, v_n\}$ of V, we can associate to every element of $g \in G$ a matrix $\pi(g)$ with coefficients in $\mathbb{C}$. Observe that every coefficient defines in this fashion a function in the group. We will call this type of functions *coefficient functions*. In the S^1 case the coefficient functions are precisely the functions $f_n(x) = e^{ln\,x}$, and hence Fourier theory says that the coefficient functions of S^1 form a Hilbert basis for $L^2(S^1)$. The Peter-Weyl theorem says that the same result is true for any compact group G. At the end of this section we will give some indication on how to get, given a compact group G, all its irreducible representations.

1 The Peter-Weyl Theorem

The Peter-Weyl Theorem is one of the most important results in representations theory. This theorem relate the representation theory of a compact group G with the space $L^2(G)$ of the square integrable functions. We will start this sections giving the basic definitions of a Lie group and a Lie algebra and the relation of a Lie group with its Lie algebra, and we will set the notation that we will use in the rest of this notes.

In the next subsection we will study the representations of a Lie group, with special emphasis to the case where G is compact. The goal will be to decompose the space $L^2(G)$ as a direct sum of irreducible representations under the left and right regular action. For this we will first define the concept of irreducible, completely reducible and unitary representation. The we will observe that if G is compact, then every irreducible representation of G is unitary and finite dimensional. Now given an irreducible representation of G, we can take the space spanned by its coefficient functions. This space is invariant and irreducible under the left and right regular action and forms an irreducible component of $L^2(G)$. The Peter-Weyl theorem will then say that the direct sum of all this components is enough to generate the space of square integrable functions.

From the Peter-Weyl theorem we see that to understand the space $L^2(G)$ is enough to classify all the irreducible representations of G. In the last subsection we will observe that if G is a compact, connected and simply connected Lie group, then its irreducible representations are in a 1-1 correspondence with the irreducible, finite dimensional representations of its Lie algebra. Besides we will observe that the Lie algebras of a compact group are reductive. Therefore we can reduce the problem of calculating the irreducible representations of compact Lie groups to the problem of finding the irreducible, finite dimensional representations of the reductive Lie algebras.

1.1 Preliminaries

Definition 1.1 *A Lie group is a group G, that also has the structure of an smooth manifold, that is compatible with multiplication and with taking inverse, i.e., the function*

$$\begin{aligned} \mu : G \times G &\longrightarrow G \\ (x, y) &\mapsto xy^{-1} \end{aligned}$$

is differentiable.

Given an element $a \in G$ we can associate to it two functions,

$$L_a : G \longrightarrow G \quad \text{and} \quad R_a : G \longrightarrow G$$
$$x \mapsto ax \qquad\qquad x \mapsto xa$$

of the group onto itself. This functions are diffeomorphism under the smooth manifold structure and are called the *left multiplication* and *right multiplication* respectively.

Definition 1.2 *A Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \mapsto \mathfrak{g}$, called the Lie bracket, with the following properties*

1. $[X,Y] = -[Y,X]$ *(Antisymmetry)*
2. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ *(Jacobi identity)*

for all $X, Y, Z \in \mathfrak{g}$.

Let G be a Lie group, and let

$$\mathfrak{g} = \mathrm{Lie}(G) := \mathfrak{X}(G)^G$$

be the space of all left invariant vector fields. This space has a natural Lie algebra structure given by the commutator of vector fields, $[X,Y] = XY - YX$, and we will call it the *Lie algebra associated to G*.

Theorem 1.3 *Given a Lie algebra $\mathfrak{g}$ there exist a unique (up to isomorphism) Lie group $\tilde{G}$ such that it is connected, simply connected and Lie$(\tilde{G}) = \mathfrak{g}$. Furthermore, if G is another connected Lie group with Lie$(G) = \mathfrak{g}$, then there exists a covering homomorphism $p : \tilde{G} \longrightarrow G$.*

1.2 Lie Group Representations

Definition 1.4 *Let G be a Lie group, and let V be a locally convex topological vector space (LCTVS). A representation of G on V is a homomorphism*

$$\pi : G \longrightarrow GL(V)$$

that is continuous in the strong topology of V, i.e., the function $(g,v) \mapsto \pi(g)v$ is continuous. In this case we say that (π, V) is a representation of G.

Example 1.5 **A)** Let $G = S^1$, and let

$$\pi : S^1 \longrightarrow GL_2(\mathbb{C})$$
$$\theta \mapsto \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Then $(\pi, \mathbb{C}^2)$ is a representation of S^1.

B) Let $G = \mathrm{SL}_n(\mathbb{R})$ and consider the representation given by assigning to every element $g \in G$ the linear transformation that it represents. This repreentation is called the *definition representation* of $G = \mathrm{SL}_n(\mathbb{R})$.

Definition 1.6 *We say that two representations (π, V), (σ, W), are equivalent, if there exists a vector space isomorphism $T : V \longrightarrow W$ that commutes with the group action, i.e, the following diagram is commutative.*

$$\begin{array}{ccc} V & \stackrel{\pi(g)}{\longrightarrow} & V \\ \Big\downarrow T & \bigcirc & \Big\downarrow T \\ W & \stackrel{\pi(g)}{\longrightarrow} & W \end{array}.$$

Example 1.7 Consider the following two representations of $\mathbb{R}^*$,

$$x \mapsto \begin{bmatrix} x & 0 \\ 0 & 1/x \end{bmatrix}, \qquad \text{y} \qquad x \mapsto \begin{bmatrix} 1/x & 0 \\ 0 & x \end{bmatrix},$$

it's clear that the two representations are equivalent.

Definition 1.8 *Let (π, V) be a representations of G. We say that $W \subset V$ is an invariant subspace if $\pi(g)W \subset W$ for all $g \in G$.*

Definition 1.9 *We say that a representation (π, V) of G is* irreducible *if the only invariant subspaces are $\{0\}$ and V.*

Lemma 1.10 (Schur's lemma) *Let (π, V) be an irreducible representation of G and let $T \in \text{End}(V)$ be an* equivariant transformation, *i.e., $\pi(g)T = T\pi(g)$, for all $g \in G$. Then $T = \lambda Id$*

Proof. To prove this lemma observe that we can always find an eigenvalue λ of T. Hence $\text{Ker}(T - \lambda Id) \neq \{0\}$. Since $\text{Ker}(T - \lambda Id)$ is an invariant subspace under the action of G and V is irreducible, we should have that $\text{Ker}(T - \lambda Id) = V$, i.e., $T = \lambda Id$. □

Definition 1.11 *We say that a representation (π, V) is* completely reducible *if there exist invariant subspaces $V_j \subset V, j = 1, \ldots, l$ such that $V_i \cap V_j = \{0\}$ and*

$$V = \oplus V_j$$

Observe that no all representations are completely reducible, for example the representation

$$\begin{aligned} \pi : \mathbb{R} &\longrightarrow GL(2, \mathbb{R}) \\ x &\mapsto \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \end{aligned}$$

is not completely reducible.

Definition 1.12 *Let H be a Hilbert space. A representation (π, H) is called* unitary *if $\pi(g)$ is a unitary operator for all $g \in G$, i.e.,*

$$\langle \pi(g)v, \pi(g)w \rangle = \langle v, w \rangle \qquad \forall g \in G.$$

Observe that if $\dim H < \infty$, then every unitary representation is completely reducible, because if $V \subset H$ is an invariant subspace, then $V^{\perp}$ is also invariant.

From now on we will assume that G is a compact group. Let (π, H) be a representation of G on the Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. Define a new inner product $(\cdot, \cdot)$ on H by the formula

$$(v, w) = \int_G \langle \pi(g)v, \pi(g)w \rangle \, dg$$

where dg is a Haar measure on G. Since G is compact, $\mathrm{Vol}(G) < \infty$ and the above integral defines a new inner product that converges for all $v, w \in H$. It is then easy to check that

$$\begin{aligned}(\pi(x)v, \pi(x)w) &= \int_G \langle \pi(gx)v, \pi(gx)w \rangle \, dg \\ &= \int_G \langle \pi(g)v, \pi(g)w \rangle \, dg = (v, w),\end{aligned}$$

i.e, (π, H) is a unitary representation with respect to this new inner product. (Here we have used that since G is compact, then it is unimodular, i.e., the left and right measures are the same).

This is the so called "unitarian trick" of Hermann Weyl. Observe that using this trick we can assume that any representation of a compat Lie group is unitary and hence any finite dimensional representation is completely reducible, in fact we also have the following result.

Theorem 1.13 *Let G be a compact group, and let (π, H) be an irreducible unitary representation of G. Then* $\dim(H) < \infty$.

Example 1.14 **A)** Let $G = S^1$. Then all the irreducible unitary representations of G are of the form $\theta \mapsto \mathrm{e}^{in\theta}, n \in \mathbb{Z}$.

B) We can define two actions of G in the space $L^2(G)$, called the *left regular action* and the *right regular action* defined, respectively, by

$$(L_a \cdot f)(x) = f(a^{-1}x) \qquad \text{y} \qquad (R_a \cdot f)(x) = f(xa)$$

If G is unimodular, then

$$\begin{aligned}\langle L_a f, L_a h \rangle &= \int_G \overline{L_a f(x)} L_a h(x) \, dx = \int_G \overline{f(a^{-1}x)} h(a^{-1}x) \, dx \\ &= \int_G \overline{f(x)} h(x) \, dx = \langle f, h \rangle \\ &= \int_G \overline{f(xa)} h(xa) \, dx = \langle R_a f, R_a h \rangle.\end{aligned}$$

Putting this two actions together we obtain a unitary representation of $G \times G$ on $L^2(G)$.

Observe that, as a Hilbert space

$$L^2(S^1) = \hat{\bigoplus_{n\in\mathbb{Z}}} \mathbb{C}\, e^{in\theta}$$

and the space generated by every $e^{in\theta}$ is invariant under the regular action, i.e., $L^2(S^1)$ is generated by the functions that we get from the irreducible representations of S^1.

In general we have the following construction. Let G be a compact group, and let (π, H) be a unitary and irreducible representation of G. Given $v, w \in H$, we define a function $c_{v,w} : G \longrightarrow \mathbb{C}$ by

$$c_{v,w}(g) = \langle \pi(g)v, w\rangle$$

the functions $c_{v,w}$ are called the *coefficient functions* of the representation.

Observation 1.15 If $(\pi, V), (\sigma, W)$ are two equivalent representations, then the space generated by its coefficient functions are the same.

There is an slightly different way of looking at the coefficient functions. Let $v, w \in H$ and define $T_{v,w} \in \mathrm{End}(H) \simeq H \otimes H^*$ by

$$T_{v,w}(u) = \langle v, u\rangle w$$

then

$$\begin{aligned} \mathrm{tr}(\pi(g)^{-1}T) &= \mathrm{Tr}(u \mapsto \langle \tilde{v}, u\rangle \pi(g^{-1})w) \\ &= \langle v, \pi(g)^{-1}w\rangle = \langle \pi(g)v, w\rangle = c_{v,w}(g) \end{aligned}$$

Theorem 1.16 *Let (π, H) be a unitary representation of G. Define an action of $G \times G$ en End(H) by $(g, h) \cdot T = \pi(g)T\pi(h)^{-1}$, and let*

$$\begin{aligned} A : End(H) &\longrightarrow L^2(G) \\ T &\mapsto (g \mapsto \mathrm{Tr}(\pi(g)^{-1}T) \end{aligned}$$

Then A is a G-equivariant linear transformation.

Definition 1.17 *Given a representation (π, H) we define the* character *of the representation to be the function $\chi(x) = Tr(\pi(x))$. Observe that this function has the property that $\chi(gxg^{-1}) = \chi(x)$.*

Theorem 1.18 (Schur's orthogonality relations) *Let H be a finite dimensional Hilbert space, and define and inner product on End(H) by $\langle T, S\rangle = \mathrm{Tr}(T^*S)$.*

A) *If $(\pi_1, H_1), (\pi_2, H_2)$ are two inequivalent irreducible representations of G. Then for all $T \in End(H_1)$, $S \in End(H_2)$*

$$\langle A(T), A(S)\rangle = 0 \qquad \text{in } L^2(G)$$

B) *If $S, T \in H$, then*

$$\langle A(S), A(T)\rangle = \frac{1}{d}\langle S, T\rangle$$

where $d = dim\ H$

Definition 1.19 *Let G be a Lie group, we define its unitary spectrum as the set*

$$\hat{G} = \left\{ \begin{array}{c} \text{equivalent classes of irreducible} \\ \text{unitary representations of } G \end{array} \right\}.$$

If we put all of this results together we see that we can build an embedding

$$\bigoplus_{\gamma \in \hat{G}} A(H_\gamma \otimes {H_\gamma}^*) \subset L^2(G)$$

that is $G \times G$ equivariant. Peter-Weyl theorem says that this embedding is actually a bijection!

Theorem 1.20

$$L^2(G) = \hat{\bigoplus_{\gamma \in \hat{G}}} A(H_\gamma \otimes {H_\gamma}^*) = \hat{\bigoplus_{\gamma \in \hat{G}}} A(End(H_\gamma))$$

Besides, if $f \in C^\infty(G)$, then $\oplus_{\gamma \in \tilde{G}} f_\gamma \to f$ uniformly, where f_γ is the projection of f to the subspace $A(End(H_\gamma))$.

To close this subsection we would like to make a couple of remarks on the Peter-Weyl theorem. We can define an action of $\mathfrak{g}$ on $C^\infty(G)$ by

$$(Xf)(g) = \left.\frac{d}{dt}\right|_{t=0} f(g \exp tX)$$

and we can extend this action to define an action of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$, that in this way gets identified with the left invariant differential operators on G. Observe that, using this identification, the space of left and right differential operators is precisely the center of the universal enveloping algebra, $Z(\mathfrak{g})$. By Schur's lemma, every $X \in Z(\mathfrak{g})$ acts as a constant on each of the irreducible components $H_\gamma \otimes {H_\gamma}^*$. Following this argument we see that if we have a differential equation on G that is bi-invariant under the group action, then we can reduce the problem of solving this differential equation to the problem of solving some related algebraic equation. Is a general principle in physics that the equations that define the fundamental forces of nature should remain invariant under the space symmetries. In the case of a Lie group G this translate to saying that we should be able to write this equations using bi-invariant differential operators. Observe that, in particular, the Casimir element of $\mathfrak{g}$ acts on G as the Laplace operator and is always in $Z(\mathfrak{g})$.

Example 1.21 In the case where $G = S^1$ we have that

$$\frac{\partial}{\partial x} \mathrm{e}^{inx} = in\, \mathrm{e}^{inx}$$

and hence

$$\frac{\partial^2}{\partial x^2} \mathrm{e}^{inx} = -n^2\, \mathrm{e}^{inx}.$$

This are the only eigenvalues of the Laplacian $\Delta = \frac{\partial^2}{\partial x^2}$

1.3 Classification of irreducible representations of compact Lie groups

Definition 1.22 *Let $\mathfrak{g}$ be a Lie algebra, and let V be a LCTVS. A representation of $\mathfrak{g}$ on V is a map*

$$\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$$

such that $\pi([X,Y]) = [\pi(X), \pi(Y)] = \pi(X)\pi(Y) - \pi(Y)\pi(X)$, and is continuous in the operator strong topology.

In a similar way to the group case, we can define the notions of irreducible, completely reducible and unitary representation. Observe that in this case a representation is called unitary if

$$\langle Xv, w\rangle + \langle v, Xw\rangle = 0 \qquad \text{for all } v, w \in V,\ X \in \mathfrak{g}$$

Theorem 1.23 *Let G be a compact connected and simply connected group, then the irreducible representations of G are in a 1-1 correspondence with the irreducible finite dimensional representations of $\mathfrak{g}$.*

If (π, H) is a finite dimensional representation of G, we will call $(d\pi, H)$ to the associated representation of $\mathfrak{g}$. This representations are related by the formulas

$$\left.\frac{d}{dt}\right|_{t=0} \pi(\exp tX) = d\pi(X).$$
$$\pi(\exp tX) = \exp(t\, d\pi(X))$$

If G is compact and connected, then $(\mathrm{Ad}, \mathfrak{g})$ is a finite dimensional representation of G and hence is completely reducible because using the "unitarian trick" we can think that is a unitary representation. From this we find that $(\mathrm{ad}, \mathfrak{g})$ is completely reducible.

Definition 1.24 *A Lie algebra $\mathfrak{g}$ is said to be* reductive *if $(\mathrm{ad}, \mathfrak{g})$ is completely reducible.*

Definition 1.25 *A group G is said to be reductive if its Lie algebra $\mathfrak{g}$ is reductive.*

Observation 1.26 If G is compact and connected, then $\mathfrak{g}$, and therefore G, are reductive.

Theorem 1.27 *If $\mathfrak{g}$ is reductive, then,*

$$\mathfrak{g} = \zeta(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$$

where $\zeta(\mathfrak{g})$ is the center of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$ si semisimple.

Let $\mathfrak{g}$ be a reductive Lie algebra, and let (π, V) be an irreducible representation of $\mathfrak{g}$. Then by Schur's lemma if $X \in \zeta(\mathfrak{g})$ then X acts as a constant, and therefore there exists $\lambda \in \zeta(\mathfrak{g})^*$ such that $\pi(X) = \lambda(X)\mathrm{Id}$ for all $X \in \zeta(\mathfrak{g})$. From this we find that V is irreducible as a representation of $[\mathfrak{g}, \mathfrak{g}]$ and hence all irreducible representations of a reductive Lie algebra are made up of an irreducible representation of $[\mathfrak{g}, \mathfrak{g}]$ and an element in $\zeta(\mathfrak{g})^*$.

Definition 1.28 *Let $\mathfrak{g}$ be a Lie algebra. We can define on $\mathfrak{g}$ a bilinear symmetric form, called the* Cartan-Killing form*, using the formula*

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)).$$

Definition 1.29 *A Lie algebra $\mathfrak{g}_u$ is said to be compact if its Cartan-Killing form is nondegenerated and negative definite.*

Theorem 1.30 ***A)*** *If G is a compact Lie group such that its Lie algebra $\mathfrak{g}_u$ is semisimple, then $\mathfrak{g}_u$ is a compact Lie algebra.*

B) *If $\mathfrak{g} = Lie(G)$ is a compact Lie algebra, then G is a compact Lie group.*

Theorem 1.31 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra, then $\mathfrak{g}$ has a compact real form, i.e., there exists a real compact Lie algebra $\mathfrak{g}_u$ such that $\mathfrak{g} = \mathfrak{g}_u \oplus i\mathfrak{g}_u$ as a real Lie algebra.*

Example 1.32 Let

$$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \{A \in M_{2\times 2}(\mathbb{C}) \,|\, \mathrm{tr}\, A = 0\} = \mathrm{Span}\{H, E, F\}$$

where

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and let

$$\mathfrak{su}_2 = \{A \in M_{2\times 2} | A^* + A = 0\} = \mathrm{Span}\left\{ \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\}.$$

Then $\mathfrak{g}_u := \mathfrak{su}_2$ is a real form of $\mathfrak{g}$, since clearly $\mathfrak{g} = \mathfrak{g}_u \oplus i\mathfrak{g}_u$. Observe that as real Lie algebras $\mathfrak{sl}_2(\mathbb{R}) = \mathrm{Span}_{\mathbb{R}}\{H, E, F\}$ and $\mathfrak{su}_2$ are not isomorphic, however their complexifications are.

From all this results we see that we have simplified the problem of finding all irreducible representations of a compact Lie group G to the problem of finding all irreducible, finite dimensional representations of a semisimple Lie algebra $\mathfrak{g}$. To complete the program established by the Peter-Weyl theorem we still need to solve the following problems:

1. **Classification of Simple Lie algebras.** Killing practically solved this problem, however he never constructed the exceptional Lie algebras, he only mentioned that they could exists. It was finally Cartan in his 1894 thesis the one that constructed all of the exceptional Lie algebras besides clarifying and simplifying Killing's work.

2. **Classify all the irreducible finite dimensional representations of semisimple Lie algebras.** Cartan and Weyl led the classification efforts during the first part of the XX century. To achieve the classification they used the system of weights and roots, and showed that this representations are in 1-1 correspondence with the set of integral dominant weights of a given Cartan subalgebra.

3. **Describe $Z(\mathfrak{g})$ and the characters associated with the irreducible representations.** Harish-Chandra's isomorphism stablish an isomorphism between $Z(\mathfrak{g})$ and $S(\mathfrak{h})^W$ the algebra of W-invariant polynomials on $\mathfrak{h}$ where $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra, and W is the corresponding Weyl group. Harish-Chandra derived this result as part of his extraordinary work on representation theory and harmonic analysis that he developed mainly between the 50's–70's.

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