Lie Algebra Cohomology

Juan Camilo Fiallo R

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Abstract

We start with a Lie Algebra $\mathfrak{g}$ over a Field K, then we construct the universal enveloping algebra $U\mathfrak{g}$ and define the cohomology groups $H^n(\mathfrak{g}, A)$ for every (left) $\mathfrak{g}$-module A, by regarding A as a $U\mathfrak{g}$-module. Then we give proofs of the Weyl Theorem and deduce the Whitehead Lemmas as corollaries.

1 Introduction

The origin of Cohomology theory of Lie Algebras lies in algebraic topology. Chevalley-Eilenberg (see [1]) have shown that the real cohomology of the underlying topological space of a compact connected Lie group is isomorphic to the real cohomology of its algebra. We are going to give cohomological proofs of the two main theorems in the theory of Lie algebras over a field of characteristic 0.
The first of these theorem is that the finite-dimensional representations of a semi-simple Lie algebra are completely reducible. The main step in that proof will be to show that the first cohomology group of a semi-simple Lie algebra with arbitrary finite-dimensional coefficient module is trivial. This is known as the first Whitehead Lemma. Then we are going to prove that every finite dimensional Lie algebra $\mathfrak{g}$ is the split extension of a semi-simple Lie algebra by the radical of $\mathfrak{g}$. The main step in the proof of this result will be to show that the second cohomology group of a semi-simple Lie algebra with arbitrary finite-dimensional coefficient module is trivial. This is known as the second Whitehead Lemma.

2 The Universal enveloping algebra

Let K be a field. A *Lie algebra* $\mathfrak{g}$ over K is a vector space over K together with a bilinear map $[\ ,\ ] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$, satisfying the following two identities

$$[x,x] = 0, \qquad x \in \mathfrak{g}$$
$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0, \ x,y,z \in \mathfrak{g}.$$

The second equation is known as the Jacobi identity. A *Lie algebra homomorphism* $f : \mathfrak{g} \to \mathfrak{h}$ is a K-linear map compatible with the Lie Bracket. A Lie *subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ is a linear subspace closed under the bracket. A Lie subalgebra $\mathfrak{h}$ is called a *Lie ideal* of $\mathfrak{g}$, if $[x,y] \in \mathfrak{h}$ for every $x \in \mathfrak{g}$ and every $y \in \mathfrak{h}$. If $\mathfrak{h}$ is a Lie ideal of $\mathfrak{g}$, then the quotient space $\mathfrak{g}/\mathfrak{h}$ inherits the Lie algebra structure from $\mathfrak{g}$.

A Lie algebra is called *abelian* if $[x,y] = 0$ for all $x,y \in \mathfrak{g}$. This is a special case, because any K-vector space can be regarded as an abelian Lie algebra. In fact, given any K-algebra Λ we can associate (functorially) a Lie Algebra $L\Lambda$ with the same underlying vector space as Λ, the Lie bracket being defined by

$$[x,y] = xy - yx.$$

In order to define a the cohomology group of a Lie algebra we need to find a left adjoint to the functor L (i.e. a functor U determined by the Hom(,) functor, going from the category of Lie Algebras to the category of Algebras). Such functor indeed exists and the image of the Lie algebra $\mathfrak{g}$ is called the *universal enveloping algebra* of $\mathfrak{g}$ and is denoted by $U\mathfrak{g}$.
Let us start with the notion of a tensor algebra over the K-vector space M.

Definition 2.1. *Denote, for $n \geq 1$, the n-fold tensor product of M by T_nM,*

$$T_nM = M \otimes_K M \otimes_K ... \otimes_K M, \ n \ times.$$

Set $T_0M = K$. Then, the tensor algebra TM is $\bigoplus_{n=0}^{\infty} T_nM$, with the multiplication induced by

$$(m_1 \otimes ... \otimes m_p) \cdot (m'_1 \otimes ... m'_q) = m_1 \otimes ... \otimes m_p \otimes m'_1 \otimes ... m'_q$$

where $m_i, m_j \in M$ for $1 \leq i \leq p$, $1 \leq j \leq q$.

Note that TM is the free K-algebra over M; More precisely: To any K-algebra Λ and any K-linear map $f : M \to \Lambda$ there exists a unique algebra homomorphism $f_0 : TM \to \Lambda$ extending f. In other words, the functor T is left-adjoint to the underlying functor to K-vector spaces which forgets the algebra structure. This is because $f_0(m_1 \otimes ... \otimes m_p)$ may and must be defined as

$$f(m_1)f(m_2)...f(m_p).$$

Definition 2.2. *Given a K-Lie algebra $\mathfrak{g}$, we define the universal enveloping algebra $U\mathfrak{g}$ of $\mathfrak{g}$ to be the quotient of the tensor algebra $T\mathfrak{g}$ by the ideal I generated by the elements of the form*

$$x \otimes y - y \otimes x - [x,y], \ x,y \in \mathfrak{g}.$$

Then $U\mathfrak{g} = T\mathfrak{g}/I$.

Clearly, we have a canonical mapping of K-vector spaces $i : \mathfrak{g} \to U\mathfrak{g}$ defined by the projection $\mathfrak{g} \subset T\mathfrak{g} \to U\mathfrak{g}$ which can be easily turned into a Lie algebra homomorphism

$$i : \mathfrak{g} \to LU\mathfrak{g}.$$

It is now easy to see that any Lie algebra homomorphism $f : \mathfrak{g} \to L\Lambda$ induces a unique homomorphism $f_1 : U\mathfrak{g} \to \Lambda$ since the homomorphism $f_0 : T\mathfrak{g} \to \Lambda$ vanishes on the ideal I. We may summarize this in the following proposition:

Proposition 2.3. *The universal enveloping algebra functor is a left adjoint to the functor L.*

The next step of our construction is to study the structure of $U\mathfrak{g}$. Let $\{e_i\}$, $i \in J$ be a K-basis of $\mathfrak{g}$ indexed by a simply-ordered set J. Let $I = \{i_1, ..., i_n\}$ denote an increasing sequence of elements in J. Then define $e_I = e_{i_1} \cdots e_{i_n} \in U\mathfrak{g}$ to be the projection of $e_{i_1} \otimes ... \otimes e_{i_n} \in T\mathfrak{g}$.

Theorem 2.4. *(Birkhoff-Witt) Let $\{e_i\}$, $i \in J$, be a K-basis of $\mathfrak{g}$. Then the elements e_I corresponding to all finite increasing sequence I (including the empty one) form a K-basis of $U\mathfrak{g}$.*

Proof. We are going to prove this in several steps. We begin defining the *index* of a monomial $e_{j_1} \otimes ... \otimes e_{j_n}$ as follows. For any $1 \leq i < k \leq n$, set

$$\eta_{ik} = \begin{cases} 0 & if \quad j_i \leq j_k \\ 1 & if \quad j_i > j_k \end{cases}$$

and define the index

$$ind(e_{j_1} \otimes ... \otimes e_{j_n}) = \sum_{i<k} \eta_{ik}.$$

Note that ind=0 if and only if $j_1 \leq ... \leq j_n$. Monomials having this property will be called *standard*. We now suppose $j_k > j_{k+1}$ and we wish to compare

$$ind(e_{j_1} \otimes ... \otimes e_{j_n}) \qquad and \tag{1}$$

$$ind(e_{j_1} \otimes ... \otimes e_{j_{k+1}} \otimes e_{j_k} \otimes ... \otimes e_{j_n}) \tag{2}$$

where the second monomial is obtained by interchanging e_k and e_{k+1}. Let η'_{ik} be denotes the $\eta' s$ for the second monomial. Then we have $\eta'_{ij} = \eta_{ij}$ if $i, j \neq k, k+1$; $\eta'_{ik} = \eta_{i,k+1}$, $\eta'_{i,k+1} = \eta_{ik}$ if $i < k$; $\eta'_{k,j} = \eta_{k+1,j}$, $\eta'_{k+1,j} = \eta_{k,j}$ if $j > k+1$ and $\eta'_{k,k+1} = 0$, $\eta'_{k,k+1} = 1$. Hence,

$$ind(e_{j_1} \otimes ... \otimes e_{j_n}) = 1 + ind(e_{j_1} \otimes ... \otimes e_{k+1} \otimes e_k \otimes ... \otimes e_{j_n}).$$

We use these remarks to the following claims.

Claim 2.5. *Every element of $T\mathfrak{g}$ is congruent mod I to a K-linear combination of 1 and standard monomials.*

Proof. It suffices to prove the statement for monomials. We order these by degree and for a given degree by the index. To prove the assertion for a monomial $e_{j_1} \otimes ... \otimes e_{j_n}$ it suffices to assume it for monomials of lower degree and for those of the same degree n which are of lower index than the given monomial. Assume the monomial is not standard and suppose $j_k > j_{k+1}$. We have,

$$e_{j_1} \otimes ... \otimes e_{j_n} = e_{j_1} \otimes ... \otimes e_{k+1} \otimes e_k \otimes ... \otimes e_{j_n} + e_{j_1} \otimes ... \otimes \left[e_{j_k}, e_{j_{k+1}}\right] \otimes ... \otimes e_{j_n}, \ (mod\ I).$$

The first term of the right-hand side is of lower index than the given monomial while the second is a linear combination of monomials of lower degree. The result follows from the induction hypothesis. □

We wish to show that the the cosets of 1 and the standard monomials are linearly independent and so form a basis for $U\mathfrak{g}$. For this purpose we introduce the vector space V_n with the basis $e_{j_1} \cdots e_{j_n}$, $i_1 \leq ... \leq i_n$ and the vector space $V = K1 \oplus V_1 \oplus V_2 \oplus ...$ The next claim will help us prove the independence we want.

Claim 2.6. *There exists a linear mapping σ of $T\mathfrak{g}$ onto V such that*

$$\sigma(1) = 1 \tag{3}$$

$$\sigma(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{i_n}) = e_{i_1} e_{i_2} \cdots e_{i_n}, \ if \ i_1 \leq ... \leq i_n \tag{4}$$

$$\sigma(e_{j_1} \otimes ... \otimes e_{j_n} - e_{j_1} \otimes e_{j_{k+1}} \otimes e_{j_k} \otimes ... \otimes e_{j_n}) = \sigma(e_{j_1} \otimes ... \otimes \left[e_{j_k}, e_{j_{k+1}}\right] \otimes ... \otimes e_{j_n}) \tag{5}$$

Proof. Set $\sigma(1) = 1$ and let $T_{n,j}\mathfrak{g}$ be the subspace of $T_n\mathfrak{g}$ spanned by the monomials of degree n and index less or equal to j. Suppose a linear mapping σ has already been defined for $K1 \oplus T_1\mathfrak{g}\mathfrak{g} \oplus T_1\mathfrak{g} \oplus ... \oplus T_{n-1}\mathfrak{g}$ satisfying equations 4 and 5 for the monomials in this space. We extend σ linearly to $K1 \oplus T_1\mathfrak{g} \oplus T_1\mathfrak{g} \oplus ... \oplus T_{n-1}\mathfrak{g} \oplus T_{n,j}\mathfrak{g}$ by requiring that $\sigma(e_{i_1} \otimes e_{i_2} \otimes ... \otimes e_{j_n}) = e_{i_1} \cdots e_{i_n}$ for the standard monomials of degree n.

Next assume thatσ has been defined for $K1 \oplus T_1\mathfrak{g} \oplus T_1\mathfrak{g} \oplus ... \oplus T_{n-1}\mathfrak{g} \oplus T_{n,i-1}\mathfrak{g}$, satisfying 4 and 5 for the monomials belonging to this space and let $e_{j_1} \otimes ... \otimes e_{j_n}$ be of index $i \geq 1$. Suppose $j_k > j_{k+1}$. Then we set

$$\sigma(e_{j_1} \otimes ... \otimes e_{j_n}) = \sigma(e_{j_1} \otimes ... \otimes e_{j_{k+1}} \otimes e_{j_k} \otimes ... \otimes e_{j_n}) + \sigma((e_{j_1} \otimes ... \otimes \left[e_{j_k}, e_{j_{k+1}}\right] \otimes ... \otimes e_{j_n}) \tag{6}$$

This makes sense since the two terms on the right are in $K1 \oplus T_1\mathfrak{g} \oplus T_1\mathfrak{g} \oplus ... \oplus T_{n-1}\mathfrak{g} \oplus T_{n,i-1}\mathfrak{g}$. We show first that equation 6 is independent of the choice of the pair (j_k, j_{k+1}), $j_k > j_{k+1}$. Let (j_l, j_{j+1}) be a second pair with $j_l > j_{l+1}$. There are essentially two cases: I. $l > k+1$, II. $l = k+1$.

I. Set $u_{j_k} = u$, $u_{j_{k+1}} = v$, $u_{j_l} = w$, $u_{j_{l+1}} = x$. Then the induction hypothesis permits us to write for the right hand side of 6

$$\begin{aligned} & \sigma(...v \otimes u ... \otimes x \otimes w ....) + \sigma(...v \otimes u \otimes ... \otimes [w,x] \otimes ...) \\ + \quad & \sigma(...[u,v] \otimes ... \otimes x \otimes w \otimes ...) + \sigma(... \otimes [u,v] \otimes ... \otimes [w,x] \otimes ...) \end{aligned}$$

If we start with (j_l, j_{l+1}) we obtain

$$\begin{aligned} & \sigma(...u \otimes v \otimes ... \otimes x \otimes w \otimes ...) + \sigma(... \otimes u \otimes v \otimes ... \otimes [w,x] \otimes ...) = \\ & \sigma(...v \otimes u ... \otimes x \otimes w ....) + \sigma(...[u,v] \otimes ... \otimes x \otimes w \otimes ...) \\ + \quad & \sigma(...v \otimes u \otimes ... \otimes [w,x] \otimes ...) + \sigma(... \otimes [u,v] \otimes ... \otimes [w,x] \otimes ...) \end{aligned}$$

This is the same as the value obtained before.

II. Set $u_{j_k} = u$, $u_{j_{k+1}} = v = u_{j_l}$, $u_{j_{l+1}} = w$. If we use the induction hypothesis we can change the right hand side of 6 to

$$\sigma(...w \otimes v \otimes u...) + \sigma(...[v,w] \otimes u...) + \sigma(...v \otimes [u,w]...) + \sigma(...[u,v] \otimes w...)$$

Similarly, if we start with

$$\sigma(...u \otimes w \otimes v...) + \sigma(...u \otimes [v,w]...)$$

we can wind up with

$$\sigma(w \otimes v \otimes u...) + \sigma(...w \otimes [u,v]...) + \sigma(...[u,w] \otimes v...) + \sigma(...u \otimes [v,w]...).$$

Hence we have to show that σ annihilates the following element of $K1 \oplus T_1\mathfrak{g} \oplus ... \oplus T_{n-1}\mathfrak{g}$:

$$(...[v,w] \oplus u...) - (...u \oplus [v,w]...) + (...v \oplus [u,w]...) - (...[u,w] \oplus v....) + (...[u,v] \oplus w...) - (...w \oplus [u,v]...). \tag{7}$$

Now, it follows easily from the properties of σ in $K1 \oplus T_1\mathfrak{g} \oplus ... \oplus T_{n-1}\mathfrak{g}$ that if $(...a \otimes b...) \in T_{n-1}\mathfrak{g}$ where $a, b \in T_1\mathfrak{g}$, then

$$\sigma(...a \otimes b...) - \sigma(...b \otimes a...) + \sigma(...[a,b]) = 0 \tag{8}$$

Hence σ applied to 7 gives

$$\sigma(...[[v,w],u]...) + \sigma(...[v,[u,w]]...) + \sigma(...[[u,v]w]...) \tag{9}$$

But this is zero because of Jacobi identity. Hence, in this case too, the right hand side of 6 is uniquely determinated. We now apply 6 to define σ for the monomials of degree n and index i. The linear extension of this mapping on $T_{n,i}\mathfrak{g}$ gives a mapping on $K1 \oplus ... \oplus T_{n_i}\mathfrak{g}$ satisfying our conditions. This completes the proof of the lemma.

□

Now we can come back to the proof of the Birkhoff-Witt Theorem. Claim 2.5 shows that every coset is a linear combination of $1+I$ and the cosets of the standard monomials. Claim 2.6 gives us a linear mapping σ of $T\mathfrak{g}$ into V satisfying equations 4 and 5. It is easy to see that every element of the ideal I is a linear combination of elements of the form

$$(e_{j_1}\otimes\ldots\otimes e_{j_n})-(e_{j_1}\otimes e_{j_{k+1}}\otimes e_{j_k}\otimes\ldots\otimes e_{j_n})-(e_{j_1}\otimes\ldots\otimes\left[e_{j_k},e_{j_{k+1}}\right]\otimes\ldots\otimes e_{j_n}).$$

Since σ maps these elements into 0, σ induces a linear mapping a linear mapping from $U\mathfrak{g}=T\mathfrak{g}/I$ to V. Since equation 4 holds, the induced mapping send the cosets of 1 and the standard monomial $e_{i_1}\otimes\ldots\otimes e_{i_n}$ into 1 and $e_{i_1}\cdots e_{i_n}$ respectively. Since these images are linearly independent in V we have the linear independence in $U\mathfrak{g}$ of the cosets of 1 and the standard monomials. This completes the proof. □

Corollary 2.7. *The mapping $i:\mathfrak{g}\to U\mathfrak{g}$ is injective and $K1\cap i(\mathfrak{g})=\emptyset$.*

Proof. If $\{e_j\}$ is a basis for $\mathfrak{g}$ over K, then $1=1+I$ and the cosets $i(e_j)=e_j+I$ are linearly independent. This implies both statements. □

Corollary2.7 implies that every Lie algebra $\mathfrak{g}$ over K is isomorphic to a Lie subalgebra of a Lie algebra of the form $L\Lambda$ for some K-algebra Λ (in this case $\Lambda=U\mathfrak{g}$)

Definition 2.8. *A left $\mathfrak{g}$-module A is a K-vector space A together with a homomorphism of Lie algebras $\rho:\mathfrak{g}\to L(End_K A)$.*

We may think of the elements of $\mathfrak{g}$ as acting on A and write $x\circ a$ for $\rho(x)a$, $x\in\mathfrak{g}$, $a\in A$ so that $x\circ a\in A$. Then A is a left $\mathfrak{g}$-module and $x\circ a$ is K-linear in x and a. Note also that by the universal property of $U\mathfrak{g}$ the map ρ induces a unique algebra homomorphism $\rho_1:U\mathfrak{g}\to End_K A$, thus making A in a left $U\mathfrak{g}$-module. Conversely, if A is a left $U\mathfrak{g}$-module, so that we have a structure map $\sigma:U\mathfrak{g}\to End_K A$, it is also a $\mathfrak{g}$-module using $\rho=\sigma\circ i$. Thus the notion of $\mathfrak{g}$–module and $U\mathfrak{g}$–module coincide.
An important phenomenon in the theory of Lie algebra is that the Lie algebra $\mathfrak{g}$ itself may be regarded as a left $\mathfrak{g}$–module. The structure map is written $ad:\mathfrak{g}\to L(End_K\mathfrak{g})$ and is defined by $ad(x)y=[x,y]$, $x,y\in\mathfrak{g}$. A $\mathfrak{g}$–module is called trivial, i.e. if $x\circ a=0$ for all $x\in\mathfrak{g}$. It follows that a trivial $\mathfrak{g}$–module is just a K-vector space. Conversely, any K-vector space may be regarded a a trivial $\mathfrak{g}$–module for any Lie algebra $\mathfrak{g}$.
The structure map of K, regarded as a trivial $\mathfrak{g}$-module, sends every $x\in\mathfrak{g}$ into zero. The associated (unique) algebra homomorphism $\epsilon:U\mathfrak{g}\to K$ is called the *augmentation* of $U\mathfrak{g}$. The kernel $I\mathfrak{g}$ of ϵ is called the *augmentation ideal* of $U\mathfrak{g}$. Note that $I\mathfrak{g}$ is just the ideal of $U\mathfrak{g}$ generated by $i(\mathfrak{g})$.

3 Definition of Cohomology

Recall that a *left resolution* of a R-module M over ring R is a long exact sequence of R-modules:

$$\cdots\to P_2\xrightarrow{\delta_2}P_1\xrightarrow{\delta_1}P_0\xrightarrow{\delta_0}M\to 0$$

We say that this resolution is *projective* if each P_i, $i=0,1,2,\ldots$ is projective; meaning that the functor $Hom(P_i,\cdot)$ preserves exact sequences. With this in mind we may define the functor Ext. If C is another R-module, apply $Hom(\cdot,C)$ to the chosen projective resolution of M, yielding

$$\cdots\leftarrow Hom(P_2,C)\xleftarrow{Hom(\delta_2,C)}Hom(P_1,C)\xleftarrow{Hom(\delta_1,C)}Hom(P_0,C)\xleftarrow{Hom(\delta_0,C)}0$$

with $Hom(B,C)$ deleted as before. The nth homology of this is $Ext^n(B,C)$. Again, even though directions are reversed (since $Hom(\cdot,C)$ is contravariant), different projective resolutions give isomorphic homology, and everything in sight is well defined.

Example 3.1. *Let $M = \mathbb{Z}/p\mathbb{Z}$ and $R = \mathbb{Z}$. As a projective resolution, use $\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to M \to 0$. The map from $\mathbb{Z}$ to $\mathbb{Z}$ is multiplication by p. Applying $Hom(\cdot, C)$ (and deleting the first term) gives*

$$\cdots 0 \leftarrow C \overset{\times p}{\longleftarrow} C \overset{0}{\leftarrow} 0,$$

so $Ext^0_{\mathbb{Z}}(M,C) \approx \{x \in C \ : \ px = 0\}$ and $Ext^1_{\mathbb{Z}}(M,C) \approx C/pC$.

Definition 3.2. *Given a Lie algebra $\mathfrak{g}$ over K and a $\mathfrak{g}$–module A, we define the n^{th} cohomology group of $\mathfrak{g}$ with coefficients in A by*

$$H^n(\mathfrak{g}, A) = Ext^n_{\mathfrak{g}}(K, A), \ n = 0, 1, ... \tag{10}$$

where K is regarded, of course, as a trivial $\mathfrak{g}$–module.

Note that each $H^n(\mathfrak{g}, A)$ is a K-vector space. We start by computing H^0 and H^1. For any $\mathfrak{g}$–module A, $H^0(\mathfrak{g}, A)$ is by definition $Hom_{\mathfrak{g}}(K, A)$. Now, an $\mathfrak{g}$–module homomorphism $\phi : K \to A$ is determined by the image of $1 \in K$, $\phi(1) = a \in A$. As K is regarded as a trivial module, we have that $0 = \phi(0) = \phi(x \circ 1) = x \circ a$ for every $x \in \mathfrak{g}$. Then $\phi(1) = a$ defines a $\mathfrak{g}$–module homomorphism if and only if $x \circ a = 0$. Consequently,

$$H^0(\mathfrak{g}, A) = \{a \in A \ : \ x \circ a = 0, \ for \ all \ x \in \mathfrak{g}\};$$

we call this the the subspace of invariant elements in A and denote it by $A^{\mathfrak{g}}$.

In order to study the first cohomology group (we keep calling it group even though it is a K-vector space) we define derivations.

Definition 3.3. *A derivation from a Lie algebra $\mathfrak{g}$ into a $\mathfrak{g}$–module A is a K-linear map $d : \mathfrak{g} \to A$ such that*

$$d\,[x, y] = x \circ d(y) + y \circ d(x).$$

We denote the K-vector space of all derivations by $Der(\mathfrak{g}, A)$.

Note that if A is a trivial $\mathfrak{g}$–module, a derivation is simply a Lie algebra homomorphism where A is regarded as an abelian Lie algebra. Also note that for $a \in A$ fixed we obtain a derivation $d_a : \mathfrak{g} \to A$ by setting $d_a(x) = x \circ a$. Derivations of this kind are called *inner*. The inner derivations in $Der(\mathfrak{g}, A)$ clearly form a K-vector subspace, which we denote bu $Ider(\mathfrak{g}, A)$.

Theorem 3.4. *The functor $Der(\mathfrak{g}, \cdot)$ is represented by the $\mathfrak{g}$-module I_g, (the augmentation ideal) that is, for any $\mathfrak{g}$–module A there is a natural isomorphism between the K-vector spaces $Der(\mathfrak{g}, A)$ and $Hom_{\mathfrak{g}}(I\mathfrak{g}, A)$.*

Proof. Given a derivation $d : \mathfrak{g} \to A$, we define a K-linear map $f'_d : T\mathfrak{g} \to A$ by sending $K = T^0\mathfrak{g} \subset T\mathfrak{g}$ into zero and $x_1 \otimes ... \otimes x_n$ into $x_1 \circ (x_2 \circ ... \circ (x_{n-1} \circ dx_n)...)$. Since d is a derivation f'_d vanishes on all the elements of the form $t \otimes (x \otimes y - y \otimes x - [x, y])$, $x, y \in \mathfrak{g}$ and $t \in T\mathfrak{g}$. Since A is a $\mathfrak{g}$–module, f'_d vanishes in all the elements of the form $t_1 \otimes (x \otimes y - y \otimes x - [x, y]) \otimes t_1$, for $x, \ y \in \mathfrak{g}$ and $t_1, \ t_2 \in T\mathfrak{g}$. Thus $f^{'}_d$ defines a map $f_d : I\mathfrak{g} \to A$, which is by definition a $\mathfrak{g}$-module homomorphism.
On the other hand, if $f : I\mathfrak{g} \to A$ is given, we extend f to $U\mathfrak{g}$ by setting $f(K) = 0$ and then we define a derivation $d_f : \mathfrak{g} \to A$ by $d_f = fi$, where $i : \mathfrak{g} \to U\mathfrak{g}$ is the canonical map.

1. We check that $f_{(d_f)} = f$. Viewing $I\mathfrak{g}$ as an $\mathfrak{g}$–module we have that $f(x_1 \otimes ... \otimes x_n) = f(x_1 \otimes ... \otimes i(x_n)) = x_1 \circ (x_2 \circ ... \circ (x_{n-1} \circ fi(x_n))) = x_1 \circ (x_2 \circ ... \circ (x_{n-1} \circ d_f(x_n))) = f_{d_f}(x_1 \otimes ... \otimes x_n)$.
2. We check that $d_{(f_d)} = d$. This is easily seen because $d_{f_d}(x) = f_d i(x) = d(x)$ for all $x \in \mathfrak{g}$.

3. And finally, we check that the map $f \mapsto d_f$ is K-linear. Just note that $d_{\lambda f_1 + f_2}(x) = (\lambda f_1 + f_2)i(x) = \lambda f_1 i(x) + f i(x) = \lambda d_{f_1}(x) + d_{f_2}(x)$ for all $x \in \mathfrak{g}$.

□

If the take the following free resolution of K

$$0 \to I\mathfrak{g} \to U\mathfrak{g} \to K \to 0,$$

then given a $\mathfrak{g}$−module A, we obtain that

$$H^1(\mathfrak{g}, A) = coker(Hom_{\mathfrak{g}}(U\mathfrak{g}, A) \to Hom_{\mathfrak{g}}(I\mathfrak{g}, A)).$$

Hence $H^1(\mathfrak{g}, A)$ is ismomorphic to the vector space of derivation from $\mathfrak{g}$ into A modulo those that arise from $\mathfrak{g}$−module homomorphism $f : U\mathfrak{g} \to A$. If $f(1_{U\mathfrak{g}}) = a$, then clearly $d_f(x) = x \circ a$, so that these are precisely the inner derivations. Note that any $a \in A$ defines an inner derivation $d_a : \mathfrak{g} \to A$ that can be lifted to a $\mathfrak{g}$−module homomorphism $f_{d_a} : U\mathfrak{g} \to A$ using the technique in Theorem 3.4, and in this case $f_{d_a}(1_{U\mathfrak{g}}) = a$.

Proposition 3.5. *$H^1(\mathfrak{g}, A) \cong Der(\mathfrak{g}, A)/Ider(\mathfrak{g}, A)$. If A is a trivial $\mathfrak{g}$−module, $H^1(\mathfrak{g}, A) \cong Hom_K(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], A)$.*

Proof. In the previous discussion we showed the first assertion, so it only remains to prove the second one. Since A is trivial, there are no nontrivial inner derivations, and a derivation $d : \mathfrak{g} \to A$ is simply a Lie algebra homomorphism, A being regarded as an abelian Lie Algebra. □

4 A resolution for K

In this section we describe a convenient resolution for the ground field K. For any K-vector space V, and $n > 0$, we define $E_n V$ to be the quotient of the n-fold tensor product of V, that is, $T_n V$, by the subspace generated by

$$x_1 \otimes x_2 \otimes ... \otimes x_n - (sign\sigma) x_{\sigma 1} \otimes x_{\sigma 2} \otimes ... \otimes x_{\sigma n},$$

for $x_1, ..., x_n \in V$, and all permutation σ of the n-symmetric group. We shall use $\langle x_1, ..., x_n \rangle$ to denote an element of $E_n V$ corresponding to $x_1 \otimes ... \otimes x_n$. Clearly we have

$$\langle x_1, ..., x_i, ..., x_j, ..., x_n \rangle = -\langle x_1, ..., x_j, ..., x_i, ..., x_n \rangle.$$

Note that $E_1 V \cong V$, and set $E_0 V = K$. Then $E_n V$ is called the n^{th} *exterior power* of V and the graded K-algebra $EV = \bigoplus_{n=0}^{\infty} E_n V$, with multiplication induced by that in TV, is called *the exterior algebra* on the vector space V.

Now let $\mathfrak{g}$ be a Lie algebra over K, and let V be the underlying vector space of $\mathfrak{g}$. Denote by C_n the $\mathfrak{g}$−module $U\mathfrak{g} \otimes_K E_n V$, $n = 0, 1, ...$ For short we shall write $u\langle x_1, ..., x_n \rangle$ for $u \otimes \langle x_1, ..., x_n \rangle$, $u \in U\mathfrak{g}$. We have the following result available in [2], page 243.

Fact 4.1. *Let $C_n = U\mathfrak{g} \otimes_K E_n V$ where V is the vector space underlying $\mathfrak{g}$, and let $d_n : C_n \to C_{n-1}$ be the $\mathfrak{g}$−module maps defined by*

$$\begin{aligned} d_n(\langle x_1, ..., x_n \rangle) &= \sum_{i=0}^{n} (-1)^{i+1} x_i \langle x_1, ..., \hat{x}_i, ..., x_n \rangle \\ &+ \sum_{1 \le i < j \le n} (-1)^{i+j} \langle [x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_n \rangle. \end{aligned}$$

then the sequence

$$\cdots \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \to C_0 \xrightarrow{\epsilon} K \to 0$$

is a $\mathfrak{g}$*–free resolution of the trivial* $\mathfrak{g}$*–module K.*

5 Semi-simple Lie Algebras

In this section we review some results from the Lie theory. In the whole of this section $\mathfrak{g}$ will denote a finite-dimensional Lie algebra over a field K of characteristic 0. Also, A will denote a finite-dimensional g–module.

Definition 5.1. *To any Lie algebra* $\mathfrak{g}$ *and any* $\mathfrak{g}$*–module A we define ans associated bilinear form* β *from* $\mathfrak{g}$ *to K as follows. Let* $\rho : \mathfrak{g} \to L(End_K A)$ *be the structure map of A. If* $x, y \in \mathfrak{g}$ *then* ρx *and* ρy *are K-linear endomorphisms of A. We define* $\beta(x, y)$ *to be the trace of the the endomorphism* $(\rho x)(\rho y)$*,*

$$\beta(x, y) = Tr((\rho x)(\rho y))$$

Clearly this is a bilinear symmetric form. Recall that if $A = \mathfrak{g}$ then the associated bilinear form is called the Killing form of $\mathfrak{g}$. Also note that the associated bilinear form is associative, i.e.

$$\beta([x, y], z) = Tr((\rho x \rho y - \rho y \rho x)\rho z) = Tr(\rho x(\rho y \rho z - \rho z \rho y)) = \beta(x, [y, z])$$

Definition 5.2. *Given a Lie algebra* $\mathfrak{g}$*, we defined its derived series* $\mathfrak{g}_0$*,* $\mathfrak{g}_1$*,... inductively by*

$$\mathfrak{g}_0 = \mathfrak{g}, \ \mathfrak{g}_{n+1} = [\mathfrak{g}_n, \mathfrak{g}_n], \ n = 0, 1, ...$$

A lie algebra is called solvable if there is an integer $n \geq 0$ *with* $\mathfrak{g}_n = \{0\}$*.*

The following is an important result of the theory of Lie algebras. It is known as the *Cartan's criterion for solvable lie algebras*. A proof is accessible in [3], page 68.

Theorem 5.3. ***(Cartan's criterion for solvability)*** *Suppose* $\mathfrak{g}$ *has finite dimensional module A such that*

1. *The kernel of the structure map* ρ *is solvable.*
2. $Tr((\rho x)^2) = 0$ *for every* $x \in [\mathfrak{g}, \mathfrak{g}]$*.*

Then $\mathfrak{g}$ *is solvable.*

Definition 5.4. *A Lie algebra* $\mathfrak{g}$ *is called semi-simple if* $\{0\}$ *is the only abelian ideal of* $\mathfrak{g}$*.*

The following is a rather deep result related with the Cartan's criterion for solvability of Lie algebras. It is a key fact for what is next.

Theorem 5.5. *Let* $\mathfrak{g}$ *be semi-simple (over a field characteristic 0), and A be a* $\mathfrak{g}$*–module. If the structure map* ρ *is injective, then the bilinear form* β *corresponding to A is non-degenerate.*

Proof. Let S be the kernel of the associate bilinear form β; that is, the set of all $z \in \mathfrak{g}$ such that $\beta(x, z) = 0$ for all $x \in \mathfrak{g}$. Since $\beta(a, [z, b]) = -\beta([ab], z)$ we see that S is an ideal of $\mathfrak{g}$. Then, by the Cartan criterion for solvability (Theorem 5.3) S is solvable because $Tr((\rho x)^2) = 0$ for every $a \in S$ and ρ is injective. Since $\mathfrak{g}$ is semi-simple, $S = \{0\}$ which implies that β is non-degenerate. □

Corollary 5.6. *The Killing form of a semi-simple Lie algebra is non-degenerate.*

Proof. The structure map $ad : \mathfrak{g} \to L(End_K\mathfrak{g})$ of the $\mathfrak{g}$–module $\mathfrak{g}$ has the center of $\mathfrak{g}$ as kernel. Since the center is an abelian ideal, is is trivial. Hence ad is injective. □

Corollary 5.7. *Let $\mathfrak{a}$ be and ideal in the semi-simple Lie algebra $\mathfrak{g}$. Then there exists an ideal $\mathfrak{b}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, as Lie algebras.*

Proof. Define $\mathfrak{b}$ to be the orthogonal complement of $\mathfrak{a}$ with respect to the killing form β. Clearly it is sufficient to show (i) that $\mathfrak{b}$ is an ideal and (ii) that $\mathfrak{a} \cap \mathfrak{b} = \{0\}$. To prove (i) let $x \in \mathfrak{g}$, $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$. We have that $\beta(a,[x,b]) = \beta([a,x],b) = \beta(a',b) = 0$, where $[a,x] = a' \in \mathfrak{a}$. Hence for all $x \in \mathfrak{g}$, $[x,b] \in \mathfrak{b}$ so that $\mathfrak{b}$ is an ideal. To prove (ii) let $x,y \in \mathfrak{a} \cap \mathfrak{b}$, $z \in \mathfrak{g}$; then $\beta([x,y],z) = \beta(x,[y,z]) = 0$, since $[y,z] \in \mathfrak{b}$ and $a \in \mathfrak{a}$. Since β is non-degenerate it follows that $[x,y] = 0$. Thus $\mathfrak{a} \cap \mathfrak{b}$ is an abelian ideal of $\mathfrak{g}$, hence trivial. □

Corollary 5.8. *If $\mathfrak{g}$ is semi-simple, then every ideal $\mathfrak{a}$ in $\mathfrak{g}$ is semi-simple also.*

Proof. Since $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ by Corollary 5.7, every ideal $\mathfrak{a}'$ in $\mathfrak{a}$ is also an ideal in $\mathfrak{g}$. In particular if $\mathfrak{a}'$ is an abelian ideal, it follows that $\mathfrak{a}' = 0$. □

We now return to the cohomology theory of Lie algebras. Recall that the ground field K is assumed to have characteristic 0.

Theorem 5.9. ***(Whitehead's Theorem)*** *Let A be a (finite-dimensional) simple module over the semi-simple Lie algebra $\mathfrak{g}$ with non-trivial $\mathfrak{g}$-action. Then $H^q(\mathfrak{g},A) = 0$ for all $q \geq 0$.*

Proof. Let the structure map $\rho : \mathfrak{g} \to L(End_K A)$ have kernel $\mathfrak{h}'$. By Corollary 5.7, $\mathfrak{h}'$ has a compliment $\mathfrak{h}$ in $\mathfrak{g}$, which is non-zero because A is non-trivial. Since $\mathfrak{h}$ is semi-simple by Corollary 5.8, and since ρ restricted to $\mathfrak{h}$ is injective, the associated bilinear form β is non-degenerate by Theorem 5.5. Note that β is the restriction to $\mathfrak{h}$ of the bilinear form on $\mathfrak{g}$ associated with ρ. By linear algebra we can choose K-bases $\{e_i\}$, $i = 1,...,m$, and $\{e'_j\}$, $j = 1,...,m$, of $\mathfrak{h}$ such that $\beta(e_i, e'_j) = \delta_{ij}$. We now prove the following assertions:

1. If $x \in \mathfrak{g}$ and if $[e_i, x] = \sum_{k=1}^m c_{ik} e_k$ and $\left[x, e'_j\right] = \sum_{l=1}^m d_{jl} e'_l$, then
$$c_{ij} = d_{ji}.$$
Proof. $\beta([e_i,x],e'_j) = \beta(\sum c_{ik}e_k, e'_j) = c_{ij}$; but $\beta([e_i,x]) = \beta(e_i,\left[x,e'_j\right]) = \beta(e_j, \sum d_{jl}e'_l) = d_{ji}$. □

2. Then element $\sum_{i=1}^m e_i e'_i \in U\mathfrak{g}$ is in the center of $U\mathfrak{g}$; hence for any $\mathfrak{g}$–module B the map $t = t_B : B \to B$ defined by $t(b) = \sum_{i=1}^m e_i \circ (e'_i \circ b)$ is a $\mathfrak{g}$–module homomorphism.

   Proof. Let $x \in \mathfrak{g}$, then
$$\begin{aligned} x\left(\sum_i e_i e'_i\right) &= \sum_i \left([x,e_i]e'_i + e_i x e'_i\right) = -\sum_{i,k} c_{ik} e_k e'_i + \sum_i e_i x e'_i = -\sum_{i,k} d_{ki} e_k e'_i + \sum_k e_k x e'_k \\ &+ -\sum_k e_k \left[x, e'_k\right] + \sum_k e_k x e'_k = \sum\left(\sum_k e_k e'_k\right) x \end{aligned}$$
□

It is clear that, if $\phi : B_1 \to B_2$ is a homomorphism of $\mathfrak{g}$–modules, then $t\phi = \phi t$.

3. Consider the resolution $... \to C_n \to C_{n-1} \to ... \to C_0$ of the Theorem 4.1. The homomorphism t_{C_n} defines a chain map τ of the chain into itself. We claim that τ is chain-homotopic ti the zero map.

   Proof. We have to find maps $\Sigma_n : C_n \to C_{n+1}$, $n = 0,1,...$ such that $d_1\Sigma_0 = \tau_0$ and $d_{n+1}\Sigma_n + \Sigma_{n-1}d_n = \tau_n$, $n \geq 1$. Define Σ_n to be the $\mathfrak{g}$–module homomorphism given by

   $$\Sigma\langle x_1,...,x_n\rangle = \sigma_{k=1}^{m} e_k\langle e'_k, x_1,....,x_n\rangle.$$

   The assertion is the proved by the following computation (k varies from 1 to m; i,j vary from 1 to n):

   $$\begin{aligned}(d_{n+1}\Sigma_n + \Sigma_{n-1}d_n)\langle x_1,...,x_n\rangle &= \sum_k e_k e_{k'}\langle x_1,...,x_n\rangle + \sum_{i,k}(-1)^i e_k x_i\langle e'_k, x_1,....,x_i,...,x_n\rangle \\ &+ \sum_{i,k}(-1)^i e_k\langle\left[e'_k, x_i\right],...,\hat{x}_i,...,x_n\rangle \\ &+ \sum_{k,i<j}(-1)^{i+j} e_k\langle\left[x_i, x_j\right], e'_k, x_1,...,\hat{x}_i,...,\hat{x}_j,...,x_n\rangle \\ &+ \sum_{i,k}(-1)^{i+1} x_i e_k\langle e'_k, x_1,...,\hat{x}_i,...,x_n\rangle \\ &+ \sum_{k,i<j}(-1)^{i+j} e_k\langle e_{k'}, \left[x_i, x_j\right],, x_1,...,\hat{x}_i,...,\hat{x}_j,...,x_n\rangle \\ &= \tau_n\langle x_1,...,x_n\rangle + \sum_{i,k}(-1)^i\langle e'_k, x_1,...,\hat{x}_i,...,x_n\rangle \\ &+ \sum_{i,k}(-1)^i e_k\langle\left[e'_k, x_i\right], x_1,...,\hat{x}_i,...,x_n\rangle\end{aligned}$$

   Using that $c_{ij} = d_{ji}$ as in assertion 1. we see that the two latter sums cancel each other, and thus assertion 3. is proved ☐

Consider now the map $t = t_A : A \to A$ and the induced map

$$t_* : H^q(\mathfrak{g}, A) \to H^q(\mathfrak{g}, A).$$

By the nature of t_A (see final remark in assertion 2.), it is clear that t_* may be computed as the map induced by $\tau : C \to C$. Hence, by assertion 3., t_* is the zero map. On the other hand $t : A \to A$ must either be an automorphism or teh zero map since A is simple, but it cannot be the zero map, because the trace of the linear transformation t equals $\sum_{i=1}^{m}\beta(e_i, e'_i) = m \neq 0$. Hence it follows that $H^q(\mathfrak{g}, A) = 0$ for all $q \geq 0$. ☐

Theorem 5.10. ***(The first Whitehead Lemma)*** *Let $\mathfrak{g}$ be a finite dimensional semi-simple Lie algebra and let A be a finite dimensional $\mathfrak{g}$–module. Then $H^1(\mathfrak{g}, A) = 0$.*

Proof. Suppose there is a $\mathfrak{g}$–module A with $H^1(\mathfrak{g}, A) \neq 0$. Then there is such a $\mathfrak{g}$–module A with minimal K-dimension. If A is not simple, then there is a proper submodule $0 \neq A' = A$. Consider $0 \to A' \to A \to A/A' \to 0$ and the associated long exact cohomology sequence

$$\cdots \to H^1(\mathfrak{g}, A') \to H^1(\mathfrak{g}, A) \to H^1(\mathfrak{g}, A/A') \to \cdots.$$

Since $dim_K A' < dim_K A$ and $dim_K A/A' < dim_K A$ is follows that

$$H^1(\mathfrak{g}, A') = H^1(\mathfrak{g}, A/A') = 0.$$

Hence $H^1(\mathfrak{g}, A) = 0$, which is a contradiction. It follows that A has to be simple. But then A has to be a trivial $\mathfrak{g}$−module by Theorem 5.9. We then have that $H^1(\mathfrak{g}, A) \cong Hom(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], A)$ by Proposition 3.5. Now consider

$$0 \to [\mathfrak{g}, \mathfrak{g}] \to \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to 0.$$

By Corollary 5.7 the ideal $[\mathfrak{g}, \mathfrak{g}]$ has a complement which plainly must be isomorphic to $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, in particular it must be abelian. Since $\mathfrak{g}$ is semi-simple, $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = 0$. Hence $H^1(\mathfrak{g}, A) \cong Hom_K(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], A) = 0$, which is a contradiction. It follows that $H^1(\mathfrak{g}, A) = 0$ for all $\mathfrak{g}$−modules A. □

Corollary 5.11. ***(Weyl)*** *Every (finite-dimensional) module A over a semi-simple Lie algebra $\mathfrak{g}$ is a direct sum of simple $\mathfrak{g}$−modules.*

Proof. Using induction on the K-dimension of A, we have only to show that every non-trivial submodule $=\neq A' \subset A$ is a direct summand in A. To that end we consider the short exact sequence

$$0 \to A' \to A \to A'' \to 0 \tag{11}$$

and the induced sequence

$$0 \to Hom_K(A'', A') \to Hom_K(A, A') \to Hom_K(A', A') \to 0, \tag{12}$$

which is exact since K is a field. We remark that each of the vector spaces in 12 is finite-dimensional and can be made into a $\mathfrak{g}$−module by the following procedure. Let B,C be $\mathfrak{g}$−modules; then $Hom_(B, C)$ is a $\mathfrak{g}$−module by $(xf)(b) = xf(b) - f(bx)$, $x \in \mathfrak{g}$, $b \in B$. With this in mind, 12 becomes an exact sequence of $\mathfrak{g}$−modules. Note that the invariant elements in $Hom_k(B, C)$ are precisely the $\mathfrak{g}$−module homomorphism from B to C. Now consider the long exact cohomology sequence arising from 12

$$0 \to H^0(\mathfrak{g}, Hom(A'', A')) \to H^0(\mathfrak{g}, Hom_K(A, A')) \to H^0(\mathfrak{g}, Hom_K(A'^{,A'})) \to H^1(\mathfrak{g}, Hom_K(A'', A') \to ... \tag{13}$$

By Theorem 5.10, $H^1(\mathfrak{g}, Hom_K(A'', A'))$ is trivial. Passing to the interpretation of H^0 as the group of invariant elements, we obtain a surjective map

$$Hom_{\mathfrak{g}}(A, A') \to Hom_K(A', A').$$

If follows that there is a $\mathfrak{g}$−module homomorphism $A \to A'$ inducing the identity in A'; hence 11 splits which is what we wanted. □

We proceed with the proof of the the *second Whitehead Lemma*. First, we need some additional notions. Let $0 \to A \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{h} \to 0$ be an extension of Lie algebras over K, with abelian kernel A. If $s : \mathfrak{h} \to \mathfrak{g}$ us a section, that is, a K-linear map such that $ps = 1_{\mathfrak{h}}$, we can define iA, and hence in A, an $\mathfrak{h}$−action structure by $x \circ ia = [sx, ia]$ for $a \in A$, $x \in \mathfrak{h}$, where [,] denotes the bracket in $\mathfrak{g}$. This $\mathfrak{h}$−action defined on A does not depend upon the choice of section s. Indeed if s' is another section then $sx - s'x$ lies in A since $p(sx - s'x) = 0$ and as A is commutative the action of $sx - s'x$ in A is trivial. This $\mathfrak{h}$−module structure on A is called the $\mathfrak{h}$−module structure *induced by the extension.*
An *extension* of $\mathfrak{h}$ by an $\mathfrak{h}$−module A is an extension of Lie algebras $0 \to A \to \mathfrak{g} \to \mathfrak{h} \to 0$, with abelian kernel, such that the given $\mathfrak{h}$−module structure in A agrees with the one induced by the extension.

We shall call two extensions $0 \to A \to \mathfrak{g} \to \mathfrak{h} \to 0$ and $0 \to A \to \mathfrak{g}' \to \mathfrak{h} \to 0$ *equivalent*, if there a Lie algebra isomorphism $f : \mathfrak{g} \to \mathfrak{g}'$ such that the diagram

$$\begin{array}{ccccc} A & \to & \mathfrak{g} & \twoheadrightarrow & \mathfrak{h} \\ \| & & \downarrow & & \| \\ A & \to & \mathfrak{g}' & \twoheadrightarrow & \mathfrak{h} \end{array} \tag{14}$$

commutes. We denote the set of equivalence classes of extensions of $\mathfrak{h}$ by A by $M(\mathfrak{h},A)$. By the above, $M(\mathfrak{h},A)$ contains at least one element, the equivalence class containing the semi-direct product $0 \to A \to A \times \mathfrak{h} \to \mathfrak{h} \to 0$. With this all we have the following fact that can be found in [2] page 238 Theorem 3.3.

Theorem 5.12. *(and definition) There is a one-to-one correspondence between $H^2(\mathfrak{h},A)$ and the set $M(\mathfrak{h},A)$ of equivalence classes of extensions of $\mathfrak{h}$ by A. The set $M(\mathfrak{h},A)$ has a natural K-vector space structure whose zero element is the class containing the semi-direct product $A \xrightarrow{i_A} A \times \mathfrak{h} \xrightarrow{p_{\mathfrak{h}}} \mathfrak{h}$ which splits by $i_{\mathfrak{h}} : \mathfrak{h} \to A \times \mathfrak{h}$. The bracket is defined by $[(a,x),(b,y)] = (x \circ b - y \circ a, [x,y])$.*

Proof. Let $0 \to A \to \mathfrak{g} \to \mathfrak{h} \to 0$ be an extension of Lie algebras over K. Let $s : \mathfrak{h} \to \mathfrak{g}$ be a section, that is, a K-linear map with $ps = Id_{\mathfrak{h}}$ (here p stands for the projection $\mathfrak{g} \to \mathfrak{h}$), so that as K-vector spaces $\mathfrak{g} = A \oplus \mathfrak{h}$. Then the Lie algebra structure of $\mathfrak{g}$ may be described by a K-bilinear function $h : \mathfrak{h} \times \mathfrak{h} \to A$ defined by $h(x,y) = [sx,sy] - s[x,y]$, $x,y \in \mathfrak{h}$, because over $A \oplus \mathfrak{h}$ we may define a bracket

$$[(a,x),(b,y)] = (x \circ b - y \circ a + h(x,y), [x,y])$$

wich coincides with the bracket in $\mathfrak{g}$ since the adjoint action of $\mathfrak{g}$ over A induces a $\mathfrak{g}$–module structure on A.
Note that h is a 2-cocycle in $Hom_{\mathfrak{h}}(\mathbf{C},A)$ where $\mathbf{C}$ is the resolution of Fact 4.1 for the Lie algebra $\mathfrak{h}$. In fact, because of the definition and the Jacobi identity we have

$$\begin{aligned} dh(x_1,x_2,x_3) &= x_1 \circ h(x_2,x_3) - x_2 \circ h(x_1,x_3) + x_3 \circ h(x_1,x_2) \\ &- h([x_1,x_2],x_3) - h([x_2,x_3],x_1) + h([x_1,x_3],x_2) \\ &= [sx_1,[sx_2,sx_3]] + [sx_2,[sx_3,sx_1]] + [sx_3,[sx_1,sx_2]] \\ &- s[x_1,[x_2,x_3]] - s[x_2,[x_3,x_1]] - s[x_3,[x_1,x_2]] = 0 \end{aligned}$$

Moreover notice that if s_1 and s_2 are two sections then we have for the associated bilinear functions that $h_1 - h_2 = d(s_1 - s_2)$, which means that h is defined up to a boundary. This comes from the fact that,

$$\begin{aligned} d(s_1 - s_2)(x,y) &= x \circ (s_1 - s_2)y - y \circ (s_1 - s_2)x - (s_1 - s_2)[x,y] \\ &= [s_1x,(s_1 - s_2)y] - [s_2x,(s_1 - s_2)y] - (s_1 - s_2)[x,y] \\ &= ([s_1x,s_1y] - s_1[x,y]) - ([s_2x,s_2y] - s_2[x,y]) \\ &= h_1(x,y) - h_2(x,y) \end{aligned}$$

where we have made use of the fact that the action does not depend on the choice of the section.
Now, let A be a $\mathfrak{h}$–module. To each 2-cocycle $h \in Z^2(\mathfrak{h},\mathfrak{A})$ we associate a Lie algebra $A \oplus_h \mathfrak{h}$ as the vector space $A \oplus \mathfrak{h}$ endowed with the Lie bracket

$$[(a,x),(b,y)] = (x \circ b - y \circ a + h(x,y), [x,y]).$$

The quotient map $p : A \oplus_h \mathfrak{h} \to \mathfrak{h}$, $(a,x) \mapsto x$ is a Lie algebra homomorphism with kernel A, hence define a A-extension of $\mathfrak{h}$. The map $x \mapsto (0,x)$ is section of p. Then we have defined a map

$$\begin{array}{rcl} Z^2(\mathfrak{h},A) & \to & M(\mathfrak{h},A) \\ h & \mapsto & A \oplus_h \mathfrak{h} \end{array}$$

By the previous discussion it is a surjective map. Now let us show that its kernel consists of boundaries $B^2(\mathfrak{h},A)$, that is, $A\oplus_h \mathfrak{h}$ and $A\oplus_{h'} \mathfrak{h}$ are isomorphic extensions if and only if $h-h'$ is a boundary. Suppose that $f : A\oplus_h \mathfrak{h} \to A\oplus_{h'} \mathfrak{h}$ is an isomorphism of extensions, then we know that $f(a,0) = (a,0)$ and that there exist a K-linear map $m_f : \mathfrak{h} \to A$ such that $f(0,x) = (m_f(x),x)$. We have the following:

$$\begin{aligned} dm_f(x,y) &= x\circ m_f(y) - y\circ m_f(x) - m_f([x,y]) \\ &= h(x,y) - h'(x,y) \end{aligned}$$

because $(h(x,y),0) + f(0,[x,y]) = (h(x,y) + m_f([x,y])) = f[(0,x),(0,y)] = [f(0,x), f(0,y)] = (x\circ m_f(y) - y\circ m_f(x) + h'(x,y),[x,y])$. This means that $h-h'$ is a boundary.
Conversely , if $h-h'$ is a boundary, there exist a K-linear map $\omega : \mathfrak{h} \to A$ such that $d\omega = h-h'$, then the map $f_\omega : A\oplus_h \mathfrak{h} \to A\oplus_{h'} \mathfrak{h}$ defined by $f_\omega(a,x) = (a+\omega(x),x)$ is an A-extension isomorphism. This implies that

$$\begin{aligned} H^2(\mathfrak{h},A) = Z^2(\mathfrak{h},A)/B^2(\mathfrak{h},A) &\to M(\mathfrak{h},A) \\ [h] &\longmapsto A\oplus_h \mathfrak{h} \end{aligned}$$

is an isomorphism. This concludes the proof. □

Now we a ready to prove the second Whitehead Lemma.

Corollary 5.13. ***(The second Whitehead Lemma)*** *Let $\mathfrak{g}$ be a semi-simple Lie algebra and let A be a (finite-dimensional) $\mathfrak{g}$–module. Then $H^2(\mathfrak{g},A) = 0$.*

Proof. Suppose there is a module A with $H^2(\mathfrak{g},A) \neq 0$. Then there is such a $\mathfrak{g}$–module with minimal K-dimension. If A is not simple, then there is a proper submodule $0 \neq A' \subset A$. Consider $0 \to A' \to A \to A/A' \to 0$ and the associated long exact sequence in cohomology

$$\cdots \to H^2(\mathfrak{g},A') \to H^2(\mathfrak{g},A) \to H^2(\mathfrak{g},A/A') \to \cdots.$$

Since A' is a proper submodule, the minimality property of A leads to the contradiction $H^2(\mathfrak{g},A) = 0$. Hence A has to be simple. But then A has to be a trivial $\mathfrak{g}$–module by Proposition 5.9. Since K is the only simple trivial $\mathfrak{g}$–module, we have to show that $H^2(\mathfrak{g},K) = 0$. This will yield the desired contradiction.
By the interpretation of H^2 given by Theorem 5.12, we have to show that every central extension

$$0 \to K \xrightarrow{i} \mathfrak{h} \xrightarrow{p} \mathfrak{g} \to 0$$

of the Lie algebra $\mathfrak{g}$ splits.
Let $s : \mathfrak{g} \to \mathfrak{h}$ be a K-linear section such that $ps = 1_\mathfrak{g}$, which exists if we regard $\mathfrak{h}$ and $\mathfrak{g}$ as vector spaces. Using the section s, we define, in the K-vector space underlying $\mathfrak{h}$, a $\mathfrak{g}$–module structure by

$$x\circ y = [sx,y],\ x\in\mathfrak{g},\ y\in\mathfrak{h}$$

where the bracket is in $\mathfrak{h}$. Note that since p is a Lie algebra homomorphism and $ps = 1_\mathfrak{g}$ we have that $s([x,x'])-[sx,sx'] \in iK$. So that for $x,x' \in \mathfrak{g}$, $y\in\mathfrak{h}$ we have $x'\circ(x\circ y) - x\circ(x'\circ y) = [s[x',x],y] + [ik,y] = [x',x]\circ y$ where $k\in K$. Clearly K is a submodule of the $\mathfrak{g}$–module $\mathfrak{h}$ so defined.
Now regard $\mathfrak{h}$ as a $\mathfrak{g}$–module. By Corollary 5.11 K is a direct summand in $\mathfrak{h}$, say $\mathfrak{h} = K\oplus\mathfrak{h}'$. If $x,x' \in \mathfrak{h}'$ and $k,k' \in K$ then $[x'+k', x+k] = [x',x]$ which means that $[x',x] \in \mathfrak{h}'$. Hence $\mathfrak{h}'$ is a Lie subalgebra of $\mathfrak{h}$ and must be isomorphic to $\mathfrak{g}$. Consequently, $\mathfrak{h} \cong K\oplus\mathfrak{g}$ and the central extension of we proposed above splits as we wanted. □

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