# The CSR Function

(Continued Square Root)

The function defined as follows:

$$\mathbf{A} = \sqrt{a_1 + \sqrt{a_2 + \sqrt{a_3 + \sqrt{a_4 + \sqrt{a_5 + \sqrt{a_6 + \sqrt{a_7 + \cdots}}}}}}}$$

(see PC34-10) provides some interesting computations, using nothing more advanced than square root and addition. The accompanying table shows the value for A for various sequences of a's, calculated by Herman P. Robinson on his Wang 720C.

| | | |
|---|---|---|
| Consecutive integers | A72449 | 1.75793 27566 18004 53270 88196 38218 13852 76531 99922 |
| Even integers | A257574 | 2.15847 68723 11039 76565 58534 79807 02524 16696 94440 |
| Odd integers | A257575 | 1.85025 31288 25914 28891 21451 97359 99374 79416 17995 |
| Fibonacci | A105817 | 1.66198 24623 27811 55796 76060 81815 13129 50561 67562 |
| Powers of 2 | A257576 | 1.78316 58092 64098 88271 03049 92255 00328 85836 79511 |
| Squares | A99879 | 1.94265 54227 63987 32822 14132 91412 66723 76880 73630 |
| Cubes | A257577 | 2.17678 85971 33441 98583 98570 61721 83836 53483 24865 |
| 4th powers | A257578 | 2.46740 45317 17793 10674 79906 49563 77992 98547 96392 |
| 5th powers | A257579 | 2.82348 15128 34203 37757 13444 04555 84089 68438 53644 |
| 10th powers | A257580 | 6.05181 33295 21526 60210 47308 44812 93383 74146 88353 |
| all ones | A1622 = $\phi$ | 1.61803 39887 49894 84820 45868 34365 63811 77203 09180 |
| 1,4,2,8,5,7 repeating | A257581 | 1.87349 51093 71315 48791 93475 30993 64755 34321 31036 |
| Primes | A105546 | 2.10359 74963 39897 26261 99396 49685 32544 40421 62288 |

Aaron Herschfeld, in the American Mathematical Monthly, Vol. 42, 1935, pp. 419-429, showed that a sequence will converge if

$$\lim_{n\to\infty} (\ln \ln a_n - n \cdot \ln 2) < +\infty$$

Thus, a sequence with terms increasing no faster than $x^{2^n}$ will converge.
If the terms are exactly $x^{2^n}$, the CSR is $x\tau$, where $\tau$ is the golden mean. The sequence is started with $n = 1$.

Mr. Robinson points out the paradoxical situation that as the terms grow faster in size, fewer terms are needed for convergence, in general. An example illustrates this:

Let a given term be $10^{100}$. The following term will be $k \cdot 10^{100}$, but the square root will be $\sqrt{k} \cdot 10^{50}$. Dropping this term introduces an error of the order of $\sqrt{k}$ parts in $10^{50}$, if k is not too large. Successive square roots further reduce the error drastically. In the case of the factorials, starting with n = 17 gives an answer with a fractional error of the order of $2 \cdot 10^{-51}$. Robinson estimates that starting the factorial calculation with 35! will give a result good to more than 290 decimal places. □

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|---|---|
| Log 35 | 1.5440680443502756354984773638681431667153825148618 57 |
| ln 35 | 3.5553480614894136797061120766693673691626860838503 79 |
| $\sqrt{35}$ | 5.9160797830996160425673282915616170484155012307943 40 |
| $\sqrt[3]{35}$ | 3.2710663101885897282248069023925313440989031477789 06 |
| $\sqrt[5]{35}$ | 2.0361680046403980173608741641453176942618161675785 35 |
| $\sqrt[10]{35}$ | 1.4269435884576509835900498616503042888711259299772 05 |
| $\sqrt[100]{35}$ | 1.0361930628883961535701561250801923017400868571888 21 |
| $e^{35}$ | 1586013452313430.72812964462577466012517620395013452615426669702245280120462692325164 1062 |
| $\pi^{35}$ | 251330702007364298.61608894709750065699832912458876460025367203565715860909081425139385 |
| $\tan^{-1} 35$ | 1.5422326689561366247591507227065133543931765997729 52 |