SCAN

A5712

Guy letter

etc

87-05-13

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5726
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87-05-13

5713 ≡ 574
5581
1919
A27907

Neil J.A. Sloane,
A T & T Bell Laboratories, Room 2C-376,
600 Mountain Avenue,
MURRAY HILL, NJ 07974.

Dear Neil,

I've checked very few references, but there seem to be some remarkable omissions, not only from Sloane, but from some odd corners of combinatorics. Olga Taussky asked a question, at the last W#θ conference (Tucson), about (the largest and smallest prime divisor of) trinomial coefficients. Dick Lehmer said Euler studied them. You can make a right angled isosceles triangle in place of the usual equilateral Pascal job:

Δ A27907

								1								
							1	1	1							
						1	2	3	2	1						
					1	3	6	7	6	3	1					
				1	4	10	16	19	16	10	4	1				
			1	5	15	30	45	51	45	30	15	5	1			
		1	6	21	50	90	126	141	126	90	50	21	6	1		
	1	7	28	77	161	266	357	393	357	266	161	77	28	7	1	
1	8	36	112	266	504	784	1016	1107	1016	784	504	266	112	36	8	1

$T(n,k)$ is the coefficient of x^k in the expansion of $(1 + x + x^2)^n$

dule dued

$T(n,1)$ is Sloane 173, $T(n,2)$ is Sloane 1002 and $T(n,5)$ is Sloane 1219.

Remarkable that $T(n,3)$, $T(n,4)$, $T(n,6)$, $T(n,7)$, ... aren't there.

Sloane 1219 doesn't go very far. Chasing back your reference, JCθ $\underset{\sim}{1}$ (1966) 372, I turn back to p.356, formula (3.3) and discover that $T(n,3k-1) = Q_{3,n-1}(k)$.

But what of the other two-thirds of the sequences?

5581

Here are the gory details for incorporation in the next edition:

$T(n,3)$: 0 2 7 16 30 50 77 112 156 210 275 352 442 546 665 800 952 1122 1311 1520 1750 2002 2277 2576 2900 3250 3627 4032 4466 4930 5425 5952 6512 7106 7735 ....

$T(n,4)$: 1 6 19 45 90 161 266 414 615 880 1221 1651 2184 2835 3620 4556 5661 6954 8455 10185 12166 14421 16974 19850 23075 26676 30681 35119 40020 45415 51336 5712

$T(n,5)$: 0 3 16 51 126 266 504 882 1452 2277 3432 5005 7098 9828 13328 17748 23256 30039 38304 48279 60214 74382 .... 5713 ≡ 574

$T(n,6)$: 1 10 45 141 357 784 1554 2850 4917 8074 12727 19383 28665 41328 58276 80580 109497 146490 5714

$T(n,7)$: 0 4 30 126 393 1016 2304 4740 9042 16236 27742 45474 71955 110448 165104 241128 344964 484500 5715

$T(n,8)$: 1 15 90 357 1107 2907 6765 14355 28314 52624 93093 157950 258570 410346 633726 5716

I'll enclose a sheet on which there are some formulas. There are evidently connexions with Simon Newcomb's problem and they are not too far from Stirling numbers of the second kind. The central coefficient, $T(n,n)$, is interesting. (I'm at home, so I haven't checked if it's in Sloane : yes it's #1070). Its differences involve the central binomial coefficients, 2,6,20,70,252,..., perhaps not very surprisingly:

1 1 3 7 19 51 141 393 1107 3139 8953 25653 ... A2426

0 2 4 12 32 90 252 714 2032 5814 16700 ...

2 2 8 20 58 162 462 1318 3782 10886 ...

0 6 12 38 104 300 856 2464 7104 ...

6 6 26 66 196 556 1608 4640 ...

0 20 40 130 360 1052 3032 ...

20 20 90 230 692 1980 ...

0 70 140 462 1288 ...

70 70 322 826 ...

0 252 504 ...

252 252 ...

0 ...

Of course, the first differences are $2T(n,n-1)$. You don't have $T(n,n-1)$:

1, 2, 6, 16, 45, 126, 357, 1016, 2907, 8350, 24068, ... S717

I haven't done more than glance at the Carlitz, Roselle, Scoville paper, but presumably there's a general formula, like (3.3), but not jumping in threes (oh yes, it does):

$$T(n,k) = \binom{n+k-1}{k} - \binom{n}{1}\binom{n+k-4}{k-3} + \binom{n}{2}\binom{n+k-7}{k-6} - \dots$$

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Continuing on to the quadrinomial coefficients, we have $Q(n,0) = 1$, $Q(n,1) = n$ (S173), $Q(n,2)$ is S1002, $Q(n,3)$ is S1363, $Q(n,4)$, $Q(n,5)$, $Q(n,6)$ aren't there, but $Q(n,7)$ is S1769 and takes us back to Carlitz, Roselle & Scoville, who give every fourth one of: 1919

$$Q(n,k) = \binom{n}{0}\binom{n+k-1}{k} - \binom{n}{1}\binom{n+k-5}{k-4} + \binom{n}{2}\binom{n+k-9}{k-8} - + - \dots$$

More generally, the p-nomial coefficients, the coefficients of x^k in the expansion of $(1+x+\dots+x^{p-1})^n$ are

$$\sum_{s=0}^{\lfloor k/p \rfloor} (-1)^s \binom{n}{s}\binom{n+k-1-ps}{k-ps}, \quad k = 0,1,\dots,(p-1)n.$$

If we put $p = 2$, we should get the binomial coefficients,

$$\binom{n}{k} = \sum_{s=0}^{\lfloor k/2 \rfloor} (-1)^s \binom{n}{s}\binom{n+k-1-2s}{k-2s}$$

Formula (3.41) in Gould's Combinatorial Identities is a *bit* like that, and so are (3.56), (3.63), (3.102), $r=2$ in (3.113), (3.117) and (3.179), but I'm too lazy to try to wangle any of them into the desired form. I'll copy this to Gould and see if he can help. I'm also copying it to Jim Propp, as it's somewhat up his street. I also enclose some tables of (trinomial and) quadrinomial coefficients, many diagonals and columns of which, should be in Sloane, e.g. $Q(n,4)$, $Q(n,5)$, $Q(n,6)$, $Q(n,7)$, $Q(2n,3n)$, $Q(n,\lfloor 3n/2 \rfloor)$, perhaps $Q(2n-1,\ 3n-2)$, $Q(2n-2,\ 3n-4)$, ... and $Q(n,n)$, $Q(n,n-1)$, ... Then there's the 5-nomial coefficients, and so on.... All but a finite number are not in Sloane.

S718 2 S719 3 S720 4 1919

Best wishes,

S721 5 S724 S722 6 S723 7 S724 8

Yours sincerely,

Richard

Richard K. Guy.

RKG:1

P.S. This is all (?) in Comtet, Adv. Combin., Reidel, 1974, pp.77-78, who quotes André (1875-1873 in his list of refs?) and Montel, 1942.

Encl:

S725 9 S726 10

Sloane 173

$$1 \quad Q_{3,n-1}(1) = T(n,2) \quad \text{Sloane 1002}$$

$$1 \quad 1 \quad 1 \quad Q_{3,n-1}(2) = T(n,5) \quad \text{Sloane 1219}$$

$$1 \quad 2 \quad 3 \quad 2 \quad 1$$

$$1 \quad 3 \quad 6 \quad 7 \quad 6 \quad 3 \quad 1 \quad Q_{3,n-1}(3) = T(n,8)$$

$$1 \quad 4 \quad 10 \quad 16 \quad 19 \quad 16 \quad 10 \quad 4 \quad 1 \quad Q_{3,n-1}(4) = T(n,11)$$

$$1 \quad 5 \quad 15 \quad 30 \quad 45 \quad 51 \quad 45 \quad 30 \quad 15 \quad 5 \quad 1$$

$$1 \quad 6 \quad 21 \quad 50 \quad 90 \quad 126 \quad 141 \quad 126 \quad 90 \quad 50 \quad 21 \quad 6 \quad 1 \quad Q_{3,n-1}(k) = T(n,3k-1)$$

$$1 \quad 7 \quad 28 \quad 77 \quad 161 \quad 266 \quad 357 \quad 393 \quad 357 \quad 266 \quad 161 \quad 77 \quad 28 \quad 7 \quad 1$$

$$1 \quad 8 \quad 36 \quad 112 \quad 266 \quad 504 \quad 784 \quad 1016 \quad 1107 \quad 1016$$

$$1 \quad 9 \quad 45 \quad 156 \quad 414 \quad 882 \quad 1554 \quad 2304 \quad 2907 \quad 3139$$

$$1 \quad 10 \quad 55 \quad 210 \quad 615 \quad 1452 \quad 2850 \quad 4740 \quad 6765 \quad 8350 \quad 8953$$

$$4917 \quad 9042 \quad 14355 \quad 19855 \quad 25653$$

$$28314 \quad 43252$$

L. Carlitz, D. P. Roselle & RA Scoville, Perms & Seq^s c̄ Repet^ns by # of increases, J. Combin Theory 1 (1966) 3?? – 373

p. 356 (3.3)

Tables on p 372.

$$Q_{n,m}(k) = \sum_{s=0}^{k-1} (-1)^s \binom{m+1}{s}\binom{(k-s)n+m-1}{m}$$

$$Q_{3,n-1}(k) = \sum_{s=0}^{k} (-1)^s \binom{n}{s}\binom{3k+n-2-3s}{n-1}$$

cf Simon Newcomb's problem (Riordan)
& Stirling numbers of the 2nd kind.

$$T(n,0) = T(n,2n) = 1$$

$$T(n,1) = T(n,2n-1) = n$$

$$T(n,2) = T(n,2n-2) = \binom{n+1}{2}$$

$$T(n,3) = T(n,2n-3) = \binom{n+1}{3} + \binom{n}{2} = \frac{n(n-1)(n+4)}{3!} = \binom{n+2}{3} - n\binom{n}{0}$$

$$T(n,4) = T(n,2n-4) = \binom{n+2}{4} + \binom{n}{3} = \frac{n(n-1)(n^2+7n-6)}{4!} = \binom{n+3}{4} - n^{\cancel{4}}\binom{n}{1}$$

$$T(n,5) = T(n,2n-5) = \binom{n+2}{5} + 2\binom{n+1}{4} = \frac{(n+1)n(n-1)(n-2)(n+12)}{5!} = \binom{n+4}{5} - n\binom{n}{2}$$

$$T(n,6) = T(n,2n-6) = n(n-1)(n-2)(n^3+18n^2+17n-120)/6!$$

$$= \binom{n+5}{6} - n(n+1)(n+2) = \cdots$$

$$T(n,7) = T(n,2n-7) = n(n-1)(n-2)(n-3)(n^3+27n^2+116n-120)/7!$$

$$T(n,8) = T(n,2n-8) = (n+1)n(n-1)(n-2)(n-3)(n^3+33n^2+146n-840)/8!$$

$$T(n,k) = \binom{n}{0}\binom{n+k-1}{k} - \binom{n}{1}\binom{n+k-4}{k-3} + \binom{n}{2}\binom{n+k-7}{k-6} - + \cdots$$

$$= \sum_{s=0}^{\lfloor k/3 \rfloor} (-1)^s \binom{n}{s}\binom{n+k-3s-1}{k-3s}$$

$T(n,k)$ = coeff. of x^k in $(1+x+x^2)^n$

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																				1
																			1	1
																		1	2	3
																	1	3	6	7
																1	4	10	16	19
															1	5	15	30	45	51
														1	6	21	50	30	126	141
													1	7	28	77	161	266	357	393
												1	8	36	112	266	524	784	1016	1107
											1	9	45	156	414	882	1554	2304	2907	3139
										1	10	55	210	615	1452	2850	4740	6765	8350	8953
									1	11	66	275	880	2277	4917	9042	14355			25653
								1	12	78	352	1221	3432	8074	16236	28314				
							1	13	91	442	1651	5005	12727	27742	52624					
						1	14	105	546	2184	7098	19383	45474	93093						
					1	15	120	665	2835	9828	28665	71955	157950							
				1	16	136	800	3620	13328	41328	110448	258570								
			1	17	153	952	4556	17748	58276	165104	410346									
		1	18	171	1122	5661	23256	80580	241128	633726										
	1	19	190	1311	6954	30039	109497	344964												
1	20	210	1520	8455	38304	146490	484500													
21	231	1750	10185	48279																
253	2002	12166	60214																	
2277	14421	74382																		
16974																				

$Q(n,k)$ = coeff of x^k in $(1+x+x^2+x^3)^n$

Δ A8287

1
1 1 1
1 2 3 4
1 3 6 10 12 12
1 4 10 20 31 40 44
1 5 15 35 65 101 135 155 155
1 6 21 56 120 216 336 456 546 580
1 7 28 84 203 413 728 1128 1554 1918 2128 2128
1 8 36 120 322 728 1428 2472 3823 5328 6728 7728 8092
1 9 45 165 486 1206 2598 4950 8451 13051 18351 23607 27876 30276 30276
1 10 55 220 705 1902 4455 9240 17205 29050 44803 63460 82885 100110
1 11 66 286 990 2882 7282 16302 32802 59950 100298 154518 220198 [illegible]31258
1 12 78 364 1353 4224 11440 27456 59268 116336 209352 347568 534964 766272
1 13 91 455 1807 6019 17381 44473 102388 214500 412412 732524
14 105 560 2366 8372 25662 69680 170261 378742 773773
120 680 3045 11403 36360 106080 273975 644345