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ON LOGARITHMIC NUMBERS

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1. **Some Preliminaries.** The Logarithmic Polynomials $G_r^{(n)}(t)$ are defined by the relation

$$e^{-xt}\log(1-x^n) = -\sum_{r=1}^{\infty} G_r^{(n)}(t)\,x^r/r!\,, \quad |x|<1. \tag{1.1}$$

Whence expanding and equating the co-efficients of x^r we get

$$G_r^{(n)}(t)/r! = \sum_{j=1}^{(r/n)} (-t)^{r-jn}/(r-jn)!\,j. \tag{1.2}$$

This shows that $G_r^{(n)}(t)$ is a polynomial in t of degree $r-n$.

Again writing (1.1) in the form

$$\log(1-x^n) = -e^{xt}\sum_{r=1}^{\infty} G_r^{(n)}(t)\,x^r/r!\,, \quad |x|<1. \tag{1.3}$$

and expanding as before we obtain

$$\sum_{i=1}^{r}\binom{r}{i}G_i^{(n)}(t)\,t^i = (r-1)!\,n \text{ or } 0 \tag{1.4}$$

according as $n|r$ or $n\nmid r$.

Differentiating (1.1) with respect to t and equating the coefficients of x^r we get

$$\frac{d}{dt}G_r^{(n)}(t) = -r\,G_{r-1}^{(n)}(t). \tag{1.5}$$

so that

$$\int G_r^{(n)}(t)\,dt = -G_{r+1}^{(n)}(t)/(r+1) + \text{const} \tag{1.6}$$

i.e., the differentiation and integration of $G_r^{(n)}(t)$ respectively reduce and raise its suffix by one, along with the introduction of minus sign.

Again differentiating (1.1) with respect to x and writing

$$A_r^{(n)}(t) = [t\,G_r^{(n)}(t) + G_{r+1}^{(n)}(t)]/n \text{ for } r \geq 0. \tag{1.7}$$

where we take $G_0^{(n)}(t)=0$, and after some simplifications we get

$$\sum_{r=0}^{\infty} A_r^{(n)}(t)\, x^r/r! = x^{n-1} e^{-xt}/(1-x^n), \quad |x|<1. \tag{1.8}$$

This leads to the result

$$A_r^{(n)}(t)/r! = \sum_{j=1}^{(r+1)/n} (-t)^{r-jn+1}/(r-jn+1)\,! \tag{1.9}$$

Now from (1.1) it will be seen that

$$G_r^{(n)}(t)=0 \quad \text{for } n>r\geq 0, \tag{1.10}$$

and hence from (1.7) we get

$$A_r^{(n)}(t)=0. \quad \text{for } n>r+1\geq 1. \tag{1.11}$$

2. **Relations with known functions.** Multiplying (1.8) by $4(-1)^m x$ and summing from $m=0$ to ∞ we get

$$4\sum_{r=0}^{\infty}\sum_{m=0}^{(r/2)}(-1)^m A_r^{(2m+1)}(t)\, x^{r+1}/r! = e^{-xt}\sum_{m=0}^{\infty} 4(-1)^m x^{2m+1}/(1-x^{2m+1})$$

$$= e^{-xt}\sum_{n=1}^{\infty} r(n)\, x^n \tag{2.1}$$

where $r(n)$ is the number of representations of n as a sum of squares of two rational integers ([7], theorem 311).
It is evident from (2.1) that

$$4\sum_{m=0}^{(r/2)}(-1)^m A_r^{(2m+1)}(t) = \sum_{j=0}^{r}\frac{r!\,(-t)^j\, r(r-j+1)}{j!} \tag{2.2}$$

starting from (1.8) and making use of the following results

$$\sum_{n=1}^{\infty}\frac{\phi(n)\,x^n}{(1-x^n)} = \frac{x}{(1-x)^2}. \qquad \text{[7], Theorem (309)} \tag{2.3}$$

$$\sum_{n=1}^{\infty}\frac{\mu(n)\,x^n}{(1-x^n)} = x. \qquad \text{[7], Theorem (308)} \tag{2.4}$$

$$\sum_{n=1}^{\infty} d(n)x^n = \frac{x}{1-x}+\frac{x^2}{1-x^2}+\frac{x^3}{1-x^3}+\cdots \qquad \text{[7], Theorem (310)} \tag{2.5}$$

$$\sum_{n=1}^{\infty} \sigma(n)\, x^n = \sum_{n=1}^{\infty} \frac{n x^n}{(1-x^n)} \qquad [7].\ \text{Theorem ( )} \quad (2.6)$$

where $\phi(n)$, $\mu(n)$, $d(n)$ and $\sigma(n)$ denote respectively Euler's function, Möbius function, number of divisors of n and the sum of the divisors of n, we can prove that

$$\sum_{n=1}^{r+1} \phi(n)\, A_r^{(n)}(t) = \sum_{j=0}^{r} r!\,(-t)^j (r+1-j)/j! \text{ for every } r \geq 0. \quad (2.7)$$

$$\sum_{n=1}^{r+1} \mu(n)\, A_r^{(n)}(t) = (-t)^r \text{ for every } r \geq 0. \quad (2.8)$$

$$\sum_{n=1}^{r+1} A_r^{(n)}(t) = \sum_{j=0}^{r} \frac{r!\,(-t)^j d(r-j+1)}{j!} \text{ for every } r \geq 0. \quad (2.9)$$

and

$$\sum_{n=1}^{r+1} n\, A_r^{(n)}(t) = \sum_{j=0}^{r} \frac{\sigma(r-j+1)}{j!} (-t)^j \text{ for every } r \geq 0. \quad (2.10)$$

Multiplying (1.8) by e^{xt}, and proceeding as before, the following results can be esaily proved.

$$\sigma(r+1) = \frac{1}{r!} \sum_{n=1}^{r+1} \sum_{j=0}^{r} \binom{r}{j} n\, A_j^{(n)}(t)\, t^j. \quad (2.11)$$

$$r(r+1) = \frac{4}{r!} \sum_{m=1}^{(r/2)} \sum_{j=0}^{r} \binom{r}{j} (-1)^m A_j^{(2m+1)}(t)\, t^j \quad (2.12)$$

$$d(r+1) = \frac{1}{r!} \sum_{n=1}^{r+1} \sum_{j=0}^{r} \binom{r}{j} A_j^{(n)}(t)\, t^j. \quad (2.13)$$

It is interesting to note that results (2.11) (2.12) and (2.13) are true for all values of t, their right hand sides are functions of t, while their left hand sides are independent of t.

3. Logarithmic numbers. The values of the polynomials $G_r^{(n)}(t)$ and $A_r^{(n)}(t)$ for $t=1$ and -1 are called Logarithmic numbers. In the rest of the paper we shall be interested in the study of these numbers.

From definition we have

$$e^{-x}\log(1-x^n) = -\sum_{r=1}^{\infty} G_r^{(n)}(1)\, x^r/r! \qquad (3.1)$$

$$e^{x}\log(1-x^n) = -\sum_{r=1}^{\infty} G_r^{(n)}(-1)\, x^r/r! \qquad (3.2)$$

replacing x by $-x$ in (3.1) and taking $n=2m$ and $2m+1$, we readily get

$$(-1)^r G_r^{(2m)}(1) = G_r^{(2m)}(-1),$$

and
$$(-1)^r G_r^{(2m+1)}(1) + G_r^{(2m+1)}(-1) = G_r^{(4m+2)}(-1). \qquad (3.4)$$

In case of $A_r^{(n)}(t)$, the following results appear to be of interest.

$$\sum_{n=1}^{r+1} \mu(n)\, A_r^{(n)}(1) = (-1)^r, \quad r \geq 0. \qquad (3.5)$$

$$\sum_{n=1}^{r+1} \mu(n)\, A_r^{(n)}(-1) = 1, \; r \geq 0. \qquad (3,6)$$

In an earlier paper [5], the author has shown that

$$A_r^{(n)}(-1) = r^{(n)} A_{r-n}^{(n)}(-1) + r^{(n-1)} \qquad (3.7)$$

$$A_r^{(n)}(1) = r^{(n)} A_{r-n}^{(n)}(1) + (-1)^{r-n+1} r^{(n-1)}, \qquad (3.8)$$

where
$$r^{(n)} = r(r-1)(r-2)\cdots(r-n+1).$$

and we take $r^{(0)} = 1$. Also it can be easily proved that

$$A_r^{(2m)}(-1) = |A_r^{(2m)}(1)|. \qquad (3.9)$$

These results are of use in calculating recurssively the values of A's. We keep n fixed and go on giving values to r, starting with $r=n-1$ and taking $A_{-j}^{(n)} = 0$ for $j \geqslant 1$. The values of G's are then obtained with the help of (1.7). We give short tables of these functions for reference. We also list the values of

$$\sum_{n=1}^{r+1} A_r^{(n)}(1), \quad \sum_{n=1}^{r+1} n A_r^{(n)}(1), \quad \sum_{n=1}^{r+1} A_r^{(n)}(-1),$$

$$\sum_{n=1}^{r+1} n A_r^{(n)}(-1), \quad \sum_{n=1}^{r} G_r^{(n)}(1)/n, \quad \sum_{n=1}^{r} G_r^{(n)}(-1)/n$$

$\sum_{n=1}^{r} G_r^{(n)}(-1)$ and $\sum_{n=1}^{r} G_r^{(n)}(1)$, as these functions are found to have some interesting properties which we propose to discuss elsewhere.

4. Logarithmic numbers and some arithmetical co-efficients. From (3.1) and (3.2) it is evident that

$$\text{Cosh } x \log(1-x^n) = -\sum_{r=1}^{\infty} S_r^{(n)} x^r/r! \tag{4.1}$$

$$\text{Sinh } x \log(1-x^n) = -\sum_{r=1}^{\infty} h_r^{(n)} x^r/r!, \tag{4.2}$$

where
$$S_r^{(n)} = [G_r^{(n)}(1) + G_r^{(n)}(-1)]/2. \tag{4.3}$$

and
$$h_r^{(n)} = [G_r^{(n)}(-1) - G_r^{(n)}(1)]/2. \tag{4.4}$$

From (4.1) and (4.2) we get

$$\text{Cosh } x = \sum_{r=1}^{\infty} S_r^{(n)} x^r/r! \Big/ \sum_{r=1}^{\infty} h_r^{(n)} x^r/r! \tag{4.5}$$

Table No. 1 for $G_r^{(n)}(1)$ and allied functions

$n \backslash r$	1	2	3	4	5	6	7	8	9	10
1	1	-1	2	0	9	35	230	1624	13209	120287
2		2	-6	24	-80	450	-2142	17696	-112464	1232370
3			6	-24	60	240	-2310	9744	91224	-1134720
4				24	-120	360	-840	21840	-184464	912240
5					120	-720	2520	-6720	15120	1784160
6						720	-5040	20160	-60480	151200
7							5040	-40320	181440	-604800
8								40320	-362880	1814400
9									362880	-3628800
10										3628800

r	1	2	3	4	5	6	7	8	9	10
$\sum_{n=1}^{r} G_r^{(n)}(1)$	1	1	2	24	-11	1085	-2542	64344	-56415	4275137
$\sum_{n=1}^{r} G_r^{(n)}(1)/n$	1	0	1	10	-17	106	-437	20480	-44707	1068404

Table No. 2 for $G_r^{(n)}(-1)$ and allied functions

n \ r	1	2	3	4	5	6	7	8	9	10
1	1	3	8	24	89	415	2372	16072	125673	1112083
2		2	6	24	80	450	2142	17696	112464	1232370
3			6	24	60	480	2730	10416	151704	1285920
4				24	120	360	840	21840	184464	912240
5					120	720	2520	6720	15120	1844640
6						720	5040	20160	60480	151200
7							5040	40320	181440	604800
8								40320	362880	1814400
9									362880	3628800
10										3628800

r	1	2	3	4	5	6	7	8	9	10
$\sum_{n=1}^{r} G_r^{(n)}(-1)$	1	5	20	96	469	3135	20684	173544	1557105	16215253
$\sum_{n=1}^{r} G_r^{(n)}(-1)/n$	1	4	13	50	203	1154	6627	49352	403273	3862376

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Table No. 3 for $A_r^{(n)}(1)$ and allied functions

n \ r	0	1	2	3	4	5	6	7	8	9
1	1	0	1	2	9	44	265	1854	14833	133496
2		1	-2	9	-28	185	-846	7777	-47384	559953
3			2	-6	12	100	-690	2478	33656	-347832
4				6	-24	60	-120	5250	-40656	181944
5					24	-120	360	-840	1680	359856
6						120	-720	2520	-6720	15120
7							720	—5040	20160	-60480
8								5040	-40320	181440
9									40320	-362880
10										362880

r	0	1	2	3	4	5	6	7	8	9
$\sum_{n=1}^{r+1} n A_r^{(n)}(1)$	1	2	3	26	13	1074	-1457	61802	7929	4218722
$\sum_{n=1}^{r+1} A_r^{(n)}(1)$	1	1	1	11	-7	389	-1031	19039	-24457	1023497

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Table No. 4 for

n \ r	0	1	2	3
1	1	2	5	16
2		1	2	9
3			2	6
4				6
5				
6				
7				
8				
9				
10				

r	0	1	2
$\sum_{n=1}^{r+1} nA_r^{(n)}(-1)$	1	4	15
$\sum_{n=1}^{r+1} A_r^{(n)}(-1)$	1	3	[illegible]

Also (4.1) can be

log

Starting from (4.5) and

x cot x

tan x

Sech x

Table No. 4 for $A_r^{(n)}(-1)$ and allied functions.

$n \backslash r$	0	1	2	3	4	5	6	7	8	9	
1	1	2	5	16	65	326	1957	13700	109601	986410	← (522)
2		1	2	9	28	185	846	7777	47384	559953	— 2747 again
3			2	6	12	140	750	2562	47096	378072	
4				6	24	60	120	5250	40656	181944	
5					24	120	360	840	1680	365904	
6						120	720	2520	6720	15120	
7							720	5040	20160	60480	
8								5040	40320	181440	
9									40320	362880	
10										362880	

r	0	1	2	3	4	5	6	7	8	9	
$\sum_{n=1}^{r+1} nA_r^{(n)}(-1)$	1	4	15	76	373	2676	17539	152860	1383561	14658148	← 2750
$\sum_{n=1}^{r+1} A_r^{(n)}(-1)$	1	3	9	37	153	951	5473	42729	353937	3455083	← 2751

Also (4.1) can be written as

$$\log(1-x^n) = -\operatorname{Sech} x \sum_{r=1}^{\infty} S_r^{(n)} x^r/r! \tag{4.6}$$

Starting from (4.5) and (4.6) and using the following results

$$x \coth x = B_0 + \frac{B_2(2x)^2}{2!} + \cdots B_{2r}(2x)^{2r}(2r)/! \tag{4.7}$$

$$\tanh x = g_2 \cdot \frac{2x}{2!} + g_4 \frac{(2x)^3}{4!} + \cdots \tag{4.8}$$

$$\operatorname{Sech} x = E_0 + \frac{E_2 x^2}{2!} + \frac{E_4 x^4}{4!} + \cdots, \tag{4.9}$$

where B's, g's and E's respectively denote the numbers of Bernoulli, Genocchi and Euler, we get

$$rS^{(n)}_{r-1} \doteq (2B + h^{(n)})^r, \quad B_{2r+1} = 0, \tag{4.10}$$

$$rh^{(n)}_{r-1} \doteq (1/2)(2g + S^{(n)})^r, \quad g_{2r+1} = 0, \quad g_0 = 0. \tag{4.11}$$

$$n/r \text{ or } 0 \doteq (E + S^{(n)})^r \text{ according as } n/r \text{ or } n \nmid r, \quad E_{2r+1} = 0. \tag{4.12}$$

It is to be noted that the symbol $\doteq$ means that after expansion the power is to be replaced by a subscript and that $h_0^{(n)}$ and $S_0^{(n)} = 0$.

5. Congruence properties. In formulae (1.2) and (1.4) putting $x = 1$ and -1 respectively we get

$$G_r^{(n)}(1)/r! = \sum_{j=1}^{(r/n)} (-1)^{r-jn}/(r-jn)!\,j \tag{5.1}$$

$$G_r^{(n)}(-1)/r! = \sum_{j=1}^{(r/n)} 1/(r-jn)!\,j \tag{5.2}$$

$$G_r^{(n)}(1) - \binom{r}{1} G_{r-1}^{(n)}(1) + \cdots + \binom{r}{r-1} G_1^{(n)}(1) = n(r-1)! \tag{5.3}$$

or 0 according as n/r or $n \times r$.

$$G_r^{(n)}(-1) - \binom{r}{1} G_{r-1}^{(n)}(-1) + \cdots (-1)^{r-1} \binom{r}{r-1} G_1^{(n)}(-1)$$

$$= n(r-1)! \text{ or 0 according as } n \mid r \text{ or } n \nmid r. \tag{5.4}$$

From (5.1) and (5.2) it is evident that

$$G_n^{(n)}(1) = G_n^{(n)}(-1) = n! \tag{5.5}$$

$$G_{n+1}^{(n)}(1) = G_{n+1}^{(n)}(-1) = (n+1)! \tag{5.6}$$

and $\quad G_r^{(n)}(1)$ and $G_{r}^{(n)}(-1) = 0$ for $n \geqslant r+1. \quad (5.7)$

We now prove that

$$G_r^{(n)}(1) \equiv 0 \pmod{n!} \tag{5.8}$$

$$G_r^{(n)}(-1) \equiv 0 \pmod{n!} \tag{5.9}$$

PROOF. Since $\binom{r}{tn} = \dfrac{r}{(r-tn)!\,tn!}$ is an integer, whence $\dfrac{r!}{(r-tn)!\,t}$ is divisible by $n!$ and the congruences (5.8) and (5.9) follow from (5.1) and (5.2).

From (5.8), (5.9) and (1.7) it is evident that

$$A_r^{(n)}(1) \equiv 0\,[\bmod\,(n-1)!] \text{ and } A_r^{(n)}(-1) \equiv 0\,[\bmod\,(n-1)!]. \tag{5.10}$$

Moreover for $n > 1$

$$G_r^{(n)}(1) \equiv 0\,[\bmod\, r^{(n)}], \tag{5.11}$$

and

$$G_r^{(n)}(-1) \equiv 0\,[\bmod\, r^{(n)}], \tag{5.12}$$

while when $n = 1$ and if $r = p$ be prime then

$$G_p^{(1)}(1) \equiv 1 \pmod{p} \text{ and } G_p^{(1)}(-1) \equiv -1 \pmod{p}. \tag{5.13}$$

PROOF. We can rewrite (5.2) as

$$\begin{aligned} G_r^{(n)}(-1) &= r(r-1)\cdots(r-n+1)/1 + r(r-1)\cdots(r-2n+1)/2 \\ &\quad + r(r-1)\cdots(r-tn+1)/t + \cdots \\ &= r^{(n)}\left[\frac{1}{1} + \frac{(r-n)\cdots(r-2n+1)}{2} + \cdots \right. \\ &\quad \left. \frac{(r-n)(r-n-1)\cdots(r-tn-1)}{t}\right]\cdots, \end{aligned} \tag{5.14}$$

When $n > 1$. from (5.14) we find that each term in the brackets on its right side is an integer, since the product of t consecutive integers is divisible by $t!$, and hence the congruence (5.12) follows while when $n = 1$, from (5.14) it is evident that in general $G_r^{(1)}(-1) \not\equiv 0 \pmod{r}$. However if $n = 1$, $r = p$ be a prime then the second part of (5.13) immediately follows [Use is made of Wilson's theorem that $(p-1)! \equiv -1 \pmod{p}$]. Similarly congruence (5.11) and the first part of (5.13) can be proved.

In view of (5.11), (5.12) and (1.7) it is evident that

$$A_r^{(n)}(1) \equiv 0\,[\bmod\, r^{(n)}] \text{ and } A_r^{(n)}(-1) \equiv 0\,[\bmod\, r^{(n)}].$$

$$\text{for } n > 1,\ r > 0. \tag{5.15}$$

By elementary methods we can also prove that

$$G_{r+p}^{(n)}(1) \equiv -G_r^{(n)}(1) \pmod p \tag{5.16}$$

and

$$G_{r+p}^{(n)}(-1) \equiv G_r^{(n)}(-1) \pmod p \tag{5.17}$$

provided that $n > 1$, p is a prime $> r$ and $(r+1) > (t+1)n$, where $t = [r/n]$.

While for $n = 1$, we have

$$G_{r+p}^{(1)}(-1) - G_r^{(1)}(-1) \equiv (r+p)!/r!\,p \pmod p. \tag{5.18}$$

$$G_{r+p}^{(1)}(1) + G_r^{(1)}(1) \equiv (r+p)!/r!\,p \pmod p. \tag{5.19}$$

Of the Logarithmic numbers the following are odd

$$G_{m+1}^{(1)}(1),\ G_{4m+2}^{(1)}(1),\ G_{4m+1}^{(1)}(-1),\ G_{4m+2}^{(1)}(-1),$$

$$A_{2m}^{(1)}(1),\ A_{2m+1}^{(2)}(1),\ A_{2m}^{(1)}(-1) \text{ and } A_{2m+1}^{(2)}(-1).$$

and rest are even.

PROOF. From (5.1) we have

$$G_{4m+1}^{(1)}(1) = (4m+1) - (4m+1)4m/2 + (4m+1)\,4m\,(4m-1)/3 - \cdots \tag{5.20}$$

It is evident that all terms on the right of (5.20) are odd, except the first one which is odd and hence $G_{4m+1}^{(1)}(1)$ is odd.

Similarly other results can be proved :

Last digital properties of Logarithmic Numbers ; The last digits of most of the Logarithmic numbers are known. For example the last digits of $G_r^{(n)}(-1)$ for $n > 4$, are always 0, and the last digits of $G_r^{(2)}(-1)$, $G_r^{(3)}(-1)$ and $G_r^{(4)}(-1)$ respectively follow the order (2, 6, 4, 0, 0), (6, 4, 0, 0, 0) and (4, 0, 0, 0, 0). Their proofs consist in finding the residues modulus 10, which has been done by elementary methods, but the results and the proofs are too lengthy to be included here.

In the end it may be mentioned that the related polynomials $M_r^{(n)}(t)$ defined by

$$e^{-xt}\log(1+x^n) = \sum_{r=1}^{\infty} M_r^{(n)}(t)\,x^r/r!,\quad |x| < 1. \tag{5.21}$$

are of equal interest and may be discussed else where.

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