THE GROWTH OF DIGITAL SUMS OF POWERS OF TWO

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In this note, we give an elementary proof that $s(2^n) > \log_4 n$ for all n, where $s(n)$ denotes the sum of the digits of n written in base 10. In particular, $\lim_{n\to\infty} s(2^n) = \infty$.

The reader will notice that the lower bound is very weak. The number of digits of 2^n is $\lfloor n \log_{10} 2 \rfloor + 1$, so it is natural to conjecture that

$$\lim_{n\to\infty} \frac{s(2^n)}{n} = 4.5 \log_{10} 2.$$

However, this conjecture remains open[2].

In 1970, H. G. Senge and E. G. Strauss proved that the number of integers whose sum of digits is bounded with respect to the bases a and b is finite if and only if $\log_b a$ is rational[1]. Of course the sum of the digits of a^n in base a is 1, so this result implies that

$$\lim_{n\to\infty} s(a^n) = \infty$$

for all positive integers a except powers of 10. This work was extended by C. L. Stewart, who gave an effectively computable lower bound for $s(a^n)$ [3]. However, this lower bound is weaker than ours, and Stewart's proof relies on deep results in transcendental number theory.

We begin with two simple lemmas.

Lemma 1. *Every positive integer N can be expressed in the form*

$$N = \sum_{i=1}^{m} d[i] \cdot 10^{e[i]}$$

where $d[i]$ and $e[i]$ are integers so that $1 \leq d[i] \leq 9$ and

$$0 \leq e[1] < e[2] < \cdots < e[m]$$

Furthermore,

$$s(N) = \sum_{i=1}^{m} d[i] \geq m$$

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Proof. The proof is by strong induction on N. The case $N < 10$ is trivial. Suppose that $N \geq 10$. By the division algorithm, there exist integers $n \geq 1$ and $0 \leq r \leq 9$ so that $N = 10n + r$. By the induction hypothesis, we can express n in the form

$$n = \sum_{i=1}^{m} d[i] \cdot 10^{e[i]}$$

If $r = 0$, then

$$N = \sum_{i=1}^{m} d[i] \cdot 10^{e[i]+1}$$

and if $r > 0$ then

$$N = r \cdot 10^0 + \sum_{i=1}^{m} d[i] \cdot 10^{e[i]+1}$$

In either case, N has an expression of the required form. □

Lemma 2. *Let $2^n = A + B \cdot 10^k$ where A, B, k, n are positive integers and $A < 10^k$. Then $A \geq 2^k$.*

Proof. Since $2^n > 10^k > 2^k$, it follows that $n > k$, so 2^k divides 2^n. But 2^k also divides 10^k, therefore 2^k divides A. But $A > 0$, so $A \geq 2^k$. □

We use these lemmas to establish a lower bound on $s(2^n)$. Write

$$2^n = \sum_{i=1}^{m} d[i] \cdot 10^{e[i]}$$

so the conditions of Lemma 1 hold, and let k be an integer between 2 and m. Then $2^n = A + B \cdot 10^{e[k]}$ where

$$A = \sum_{i=1}^{k-1} d[i] \cdot 10^{e[i]}$$

and

$$B = \sum_{i=k}^{m} d[i] \cdot 10^{e[i]-e[k]}$$

Since $A < 10^{e[k]}$, Lemma 2 implies that $A \geq 2^{e[k]}$. Therefore,

$$2^{e[k]} \leq A < 10^{e[k-1]+1}$$

which implies that

$$e[k] \leq \lfloor (\log_2 10)(e[k-1]+1) \rfloor$$

We prove that $e[k] < 4^{k-1}$ for all k. It is clear that $e[1] = 0$, else 2^n would be divisible by 10. From the inequality above, we have $e[1] \leq 3$, $e[2] \leq 13$,

$e[3] \leq 46$, $e[4] \leq 156$, $e[5] \leq 521$, and $e[6] \leq 1734$. If $k \geq 7$ then $e[k-1] \geq 5$, so

$$\begin{aligned} e[k] &< (\log_2 10)e[k-1] + (\log_2 10) \\ &< \frac{10}{3}e[k-1] + \frac{10}{3} \\ &\leq \frac{10}{3}e[k-1] + \frac{2}{3}e[k-1] \\ &= 4e[k-1] \end{aligned}$$

Therefore, $e[k] < 4^{k-1}$ for all k, by induction.

We are now able to prove the main result. Note that

$$2^n < 10^{e[m]+1} \leq 10^{4^{m-1}}$$

since $10^{e[m]}$ is the leading power of 10 in the decimal expansion of 2^n.

Taking logarithms gives

$$\begin{aligned} 4^{m-1} &> n \log_{10} 2 \\ 4^{m-1} &> n/4 \\ 4^m &> n \\ m &> \log_4 n \\ s(2^n) &> \log_4 n \end{aligned}$$

hence

$$\lim_{n\to\infty} s(2^n) = \infty$$

References

[1] H. G. Senge and E. G. Straus. PV-numbers and sets of multiplicity. *Period. Math. Hungar.*, 3:93–100, 1973. Collection of articles dedicated to the memory of Alfréd Rényi, II.
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[3] C. L. Stewart. On the representation of an integer in two different bases. *J. Reine Angew. Math.*, 319:63–72, 1980.