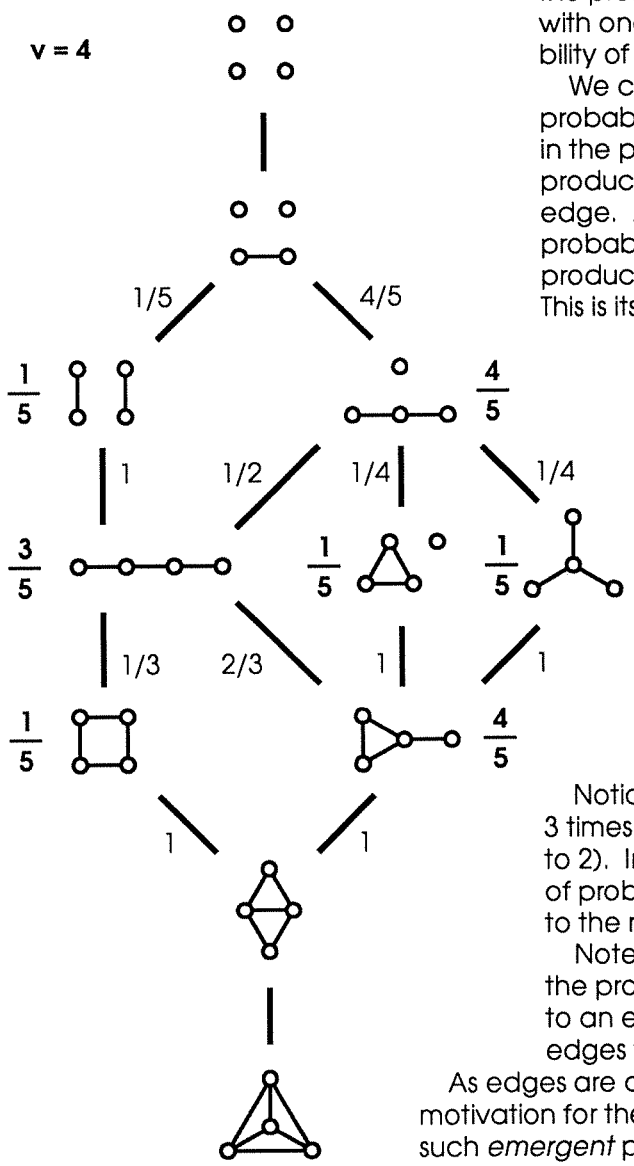
2
Probabilities of Random Graphs ($v \leq 7$)

Begin with a field of **v** isolated vertices. Choose at random one of their v-choose-2 potential edges, and draw it on the graph. The edge joins two vertices, leaving the others isolated. Choose another edge from the remaining edges. This one either shares a vertex with the first edge or it does not, so there are two possible results. The probabilities of these results simply depend on the number of remaining edges that would produce the desired graphs.

A *random graph* is defined (in this Chapter) as one that has predetermined numbers of vertices and edges and has been generated by a sequence of independent random edge additions, where at each step all remaining edges are equally likely to be chosen. Two kinds of probabilities can then be assigned to a graph, one the probability of its being generated from a graph with one-fewer edges, another the compound probability of its being generated from isolated vertices.

We can diagram these using the edge poset as a probability tree. A number assigned to a heavy edge in the poset is the probability that the lower graph is produced from the upper by the random addition of an edge. A **bold-face** number assigned to a graph is its probability of generation from scratch — the sum of products of probabilities along paths through the poset. This is its *Probability as a random graph.*

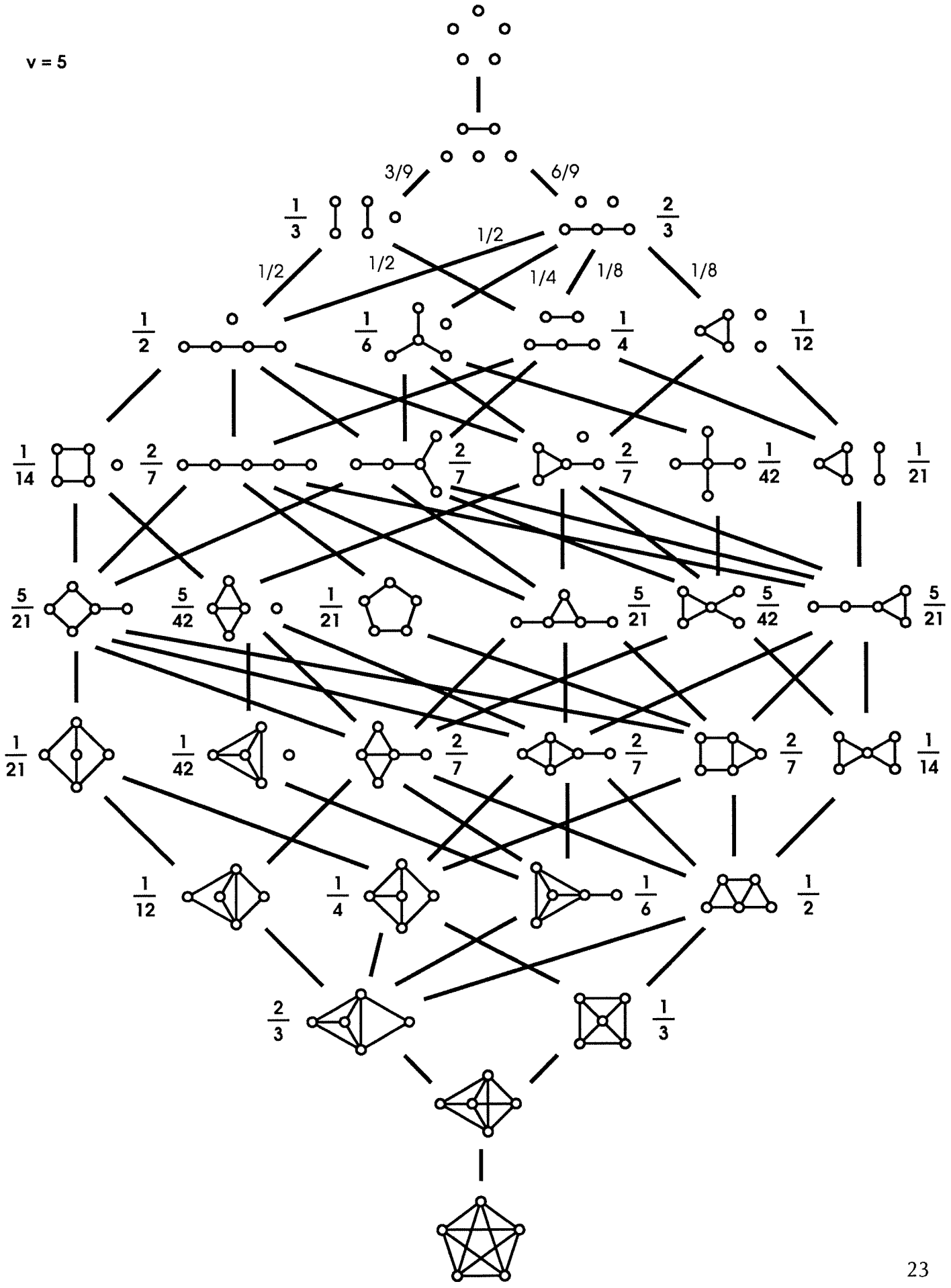


For example, on this row 3 edges have been added and the 3 possible graphs have probabilities 3/5, 1/5, and 1/5. In other words the graph at left, the path P_4, is 3 times as likely as either of the others to be generated by random edge addition.

Notice, though, that the two graphs with P = 1/5 have 3 times the symmetry as the path P_4 (6 automorphisms to 2). In fact, symmetry is inverse to probability. The ratio of probabilities of two graphs on the same row is inverse to the ratio of their orders of symmetry.

Note also that the probability of a graph is the same as the probability of its complement, since adding edges to an empty graph is structurally equivalent to deleting edges from a complete graph.

As edges are added, properties appear and change. This is the motivation for the study of Random Graph Theory, for identifying such *emergent* properties and timing their appearance.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3/9
6/9
1/3
2/3
1/2
1/2
1/2
1/4
1/8
1/8
1/2
1/6
1/4
1/12
1/14
2/7
2/7
2/7
1/42
1/21
5/21
5/42
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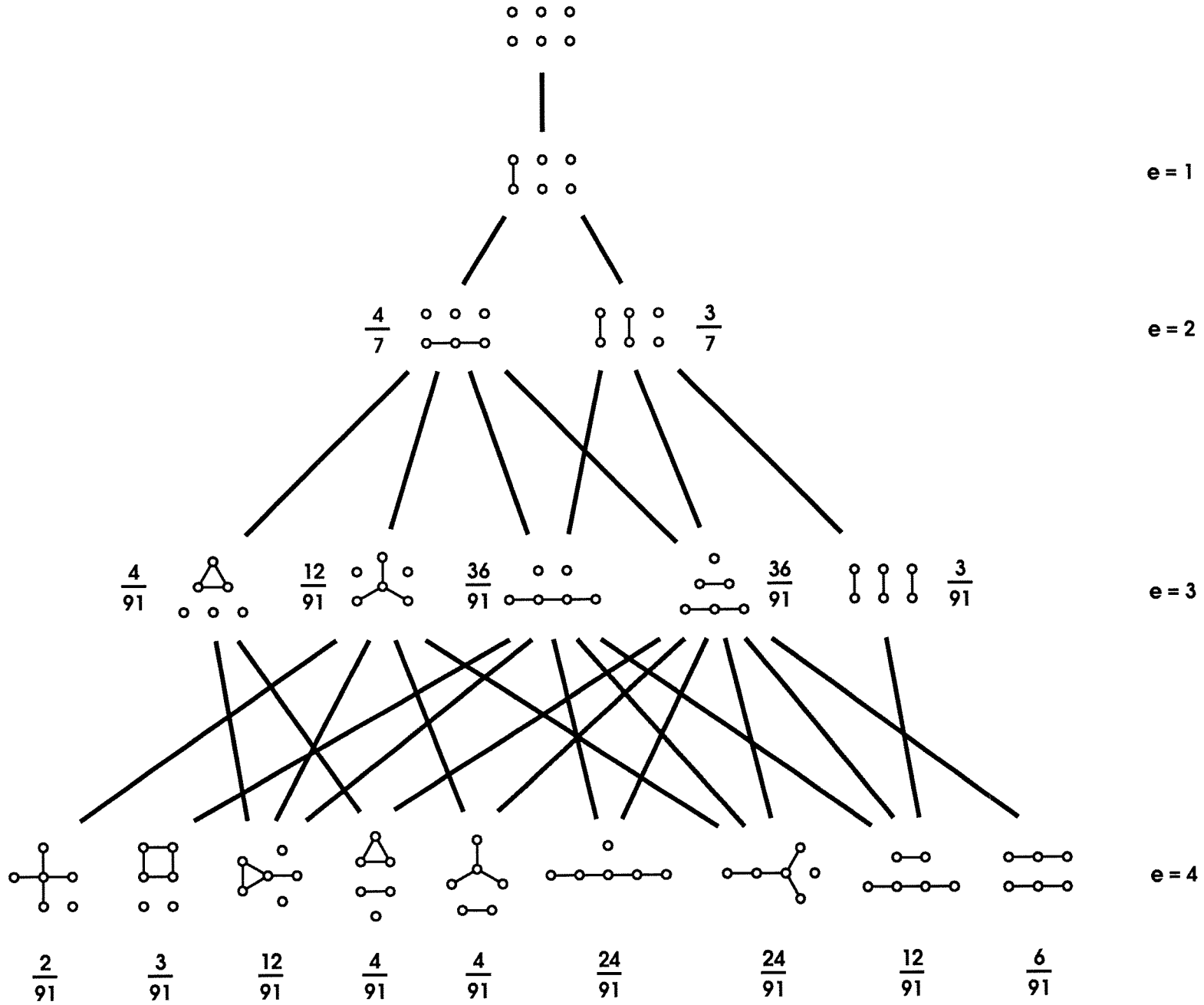
5/42
5/21
1/21
1/42
2/7
2/7
2/7
1/14
1/12
1/4
1/6
1/2
2/3
1/3

v = 6


e = 1
e = 2
e = 3
e = 4
4/7
3/7
4/91
12/91
36/91
36/91
3/91
2/91
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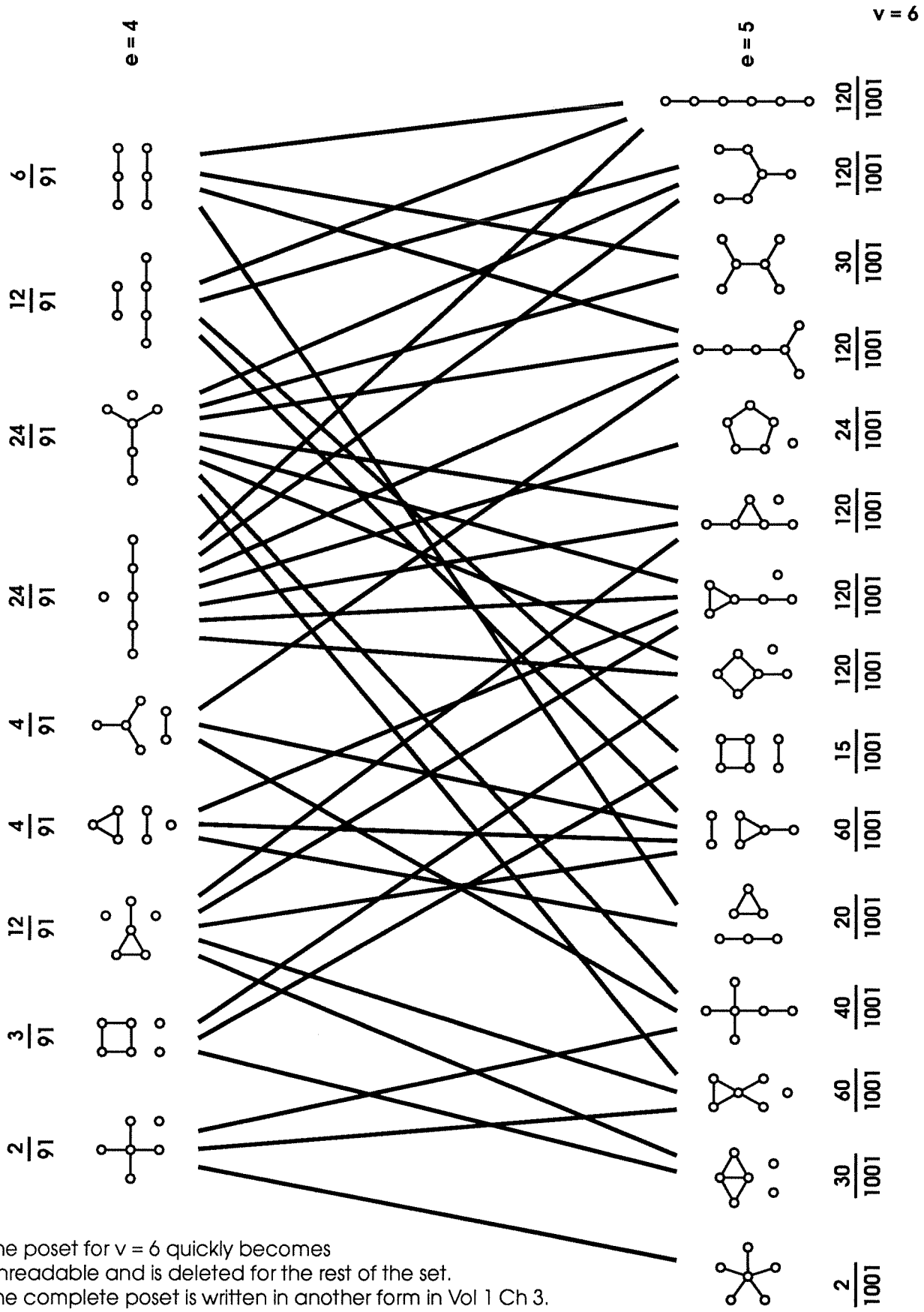
4/91
4/91
24/91
24/91
12/91
6/91



The poset for v = 6 quickly becomes
unreadable and is deleted for the rest of the set.
The complete poset is written in another form in Vol 1 Ch 3.

v = 6 continued

e = 6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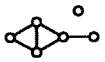
$\frac{72}{1001}$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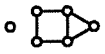
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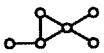
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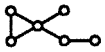
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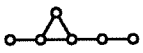
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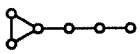
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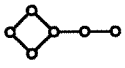
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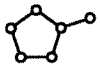
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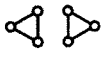
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$\frac{12}{1001}$

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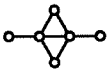
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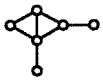
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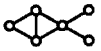
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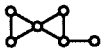
$\frac{1}{429}$


$\frac{12}{429}$


$\frac{6}{429}$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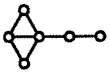
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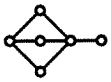
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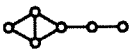
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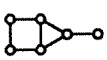
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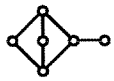
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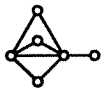 8/429

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 12/429

 12/429

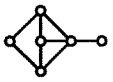
 48/429

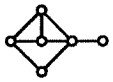
 12/429

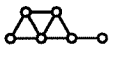
 1/429

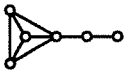
 12/429

 6/429

 24/429

 24/429

 48/429

 8/429

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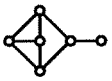
 12/429

 6/429

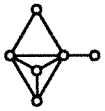
 12/429

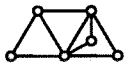
 24/429

 12/429

e = 9

 3/1001

 72/1001

 72/1001

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 12/1001

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 72/1001

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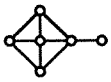
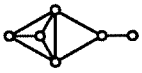
 72/1001

 36/1001

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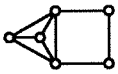
 36/1001

 72/1001

 36/1001

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 72/1001

 72/1001

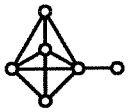
 2/1001

 12/1001

v = 6 continued

e = 10


30/1001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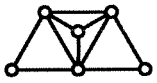
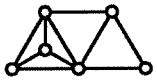

120/1001


120/1001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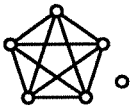
24/1001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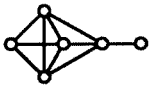
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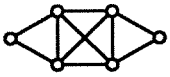
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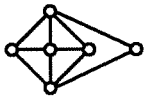
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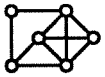
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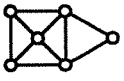
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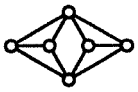
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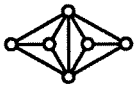
120/1001

e = 11


12/91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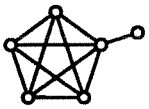
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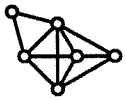
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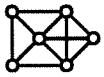
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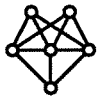
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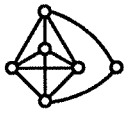
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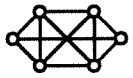
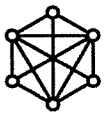
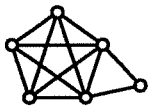
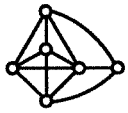

6/91

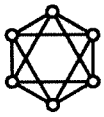
e = 12


4/91

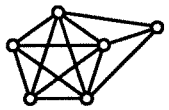

12/91

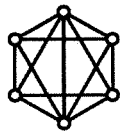

36/91


36/91

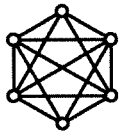

3/91

e = 13

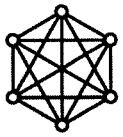

4/7


3/7

e = 14



e = 15



The numbering scheme for v = 7 is the same used in Volume 1 Chapters 1 & 3. Identification numbers are listed in complementary pairs, since a graph and its complement share the same probability.

e = 2 or 19

#3-1042	**1/2**
4-1041	**1/2**

e = 3 or 18

5-1040	**1/38**
6-1039	**4/38**
7-1038	**12/38**
8-1037	**18/38**
9-1036	**3/38**

e = 4 or 17

10-1035	**4/57**
11-1034	**1/57**
12-1033	**1/57**
13-1032	**12/57**
14-1031	**12/57**
15-1030	**2/57**
16-1029	**4/57**
17-1028	**12/57**
18-1027	**6/57**
19-1026	**3/57**

e = 5 or 16

20-1025	**10/969**
21-1024	**30/969**
22-1023	**60/969**
23-1022	**60/969**
24-1021	**60/969**
25-1020	**12/969**
26-1019	**2/969**
27-1018	**40/969**
28-1017	**30/969**
29-1016	**60/969**
30-1015	**120/969**
31-1014	**120/969**
32-1013	**120/969**
33-1012	**15/969**
34-1011	**20/969**
35-1010	**5/969**
36-1009	**60/969**
37-1008	**20/969**
38-1007	**5/969**
39-1006	**60/969**
40-1005	**60/969**

e = 6 or 15

#41-1004	**5/7752**
42-1003	**180/7752**
43-1002	**180/7752**
44-1001	**45/7752**
45-1000	**180/7752**
46-999	**30/7752**
47-998	**60/7752**
48-997	**360/7752**
49-996	**120/7752**
50-995	**360/7752**
51-994	**180/7752**
52-993	**90/7752**
53-992	**720/7752**
54-991	**180/7752**
55-990	**180/7752**
56-989	**360/7752**
57-988	**360/7752**
58-987	**360/7752**
59-986	**360/7752**
60-985	**60/7752**
61-984	**10/7752**
62-983	**1/7752**
63-982	**30/7752**
64-981	**60/7752**
65-980	**90/7752**
66-979	**120/7752**
67-978	**180/7752**
68-977	**180/7752**
69-976	**90/7752**
70-975	**360/7752**
71-974	**180/7752**
72-973	**180/7752**
73-972	**20/7752**
74-971	**180/7752**
75-970	**360/7752**
76-969	**720/7752**
77-968	**120/7752**
78-967	**60/7752**
79-966	**45/7752**
80-965	**36/7752**
81-964	**360/7752**

e = 7 or 14

#82-963	**28/7752**
83-962	**14/7752**
84-961	**84/7752**
85-960	**42/7752**
86-959	**84/7752**
87-958	**84/7752**
88-957	**336/7752**
89-956	**84/7752**
90-955	**84/7752**
91-954	**7/7752**
92-953	**42/7752**
93-952	**168/7752**
94-951	**168/7752**
95-950	**336/7752**
96-949	**56/7752**
97-948	**168/7752**
98-947	**168/7752**
99-946	**336/7752**
100-945	**84/7752**
101-944	**84/7752**
102-943	**42/7752**
103-942	**84/7752**
104-941	**168/7752**
105-940	**84/7752**
106-939	**7/7752**
107-938	**56/7752**
108-937	**42/7752**
109-936	**84/7752**
110-935	**84/7752**
111-934	**28/7752**
112-933	**84/7752**
113-932	**168/7752**
114-931	**336/7752**
115-930	**84/7752**
116-929	**28/7752**
117-928	**168/7752**
118-927	**84/7752**
119-926	**84/7752**
120-925	**168/7752**
121-924	**168/7752**
122-923	**168/7752**
123-922	**21/7752**
124-921	**168/7752**
125-920	**84/7752**
126-919	**168/7752**
127-918	**84/7752**
128-917	**42/7752**

v = 7

e = 7 or 14

#129-916	**84/7752**
130-915	**14/7752**
131-914	**336/7752**
132-913	**168/7752**
133-912	**84/7752**
134-911	**168/7752**
135-910	**336/7752**
136-909	**168/7752**
137-908	**84/7752**
138-907	**168/7752**
139-906	**168/7752**
140-905	**28/7752**
141-904	**168/7752**
142-903	**168/7752**
143-902	**168/7752**
144-901	**168/7752**
145-900	**7/7752**
146-899	**24/7752**

e = 8 or 13

147-898	**6/1938**
148-897	**3/1938**
149-896	**4/1938**
150-895	**12/1938**
151-894	**8/1938**
152-893	**24/1938**
153-892	**48/1938**
154-891	**12/1938**
155-890	**48/1938**
156-889	**8/1938**
157-888	**24/1938**
158-887	**24/1938**
159-886	**12/1938**
160-885	**12/1938**
161-884	**12/1938**
162-883	**1/1938**
163-882	**12/1938**
164-881	**48/1938**
165-880	**24/1938**
166-879	**24/1938**
167-878	**12/1938**
168-877	**6/1938**
169-876	**24/1938**
170-875	**12/1938**
171-874	**4/1938**
172-873	**12/1938**
173-872	**24/1938**
174-871	**4/1938**
#175-870	**24/1938**
176-869	**24/1938**
177-868	**4/1938**
178-867	**24/1938**
179-866	**12/1938**
180-865	**3/1938**
181-864	**24/1938**
182-863	**24/1938**
183-862	**24/1938**
184-861	**4/1938**
185-860	**2/1938**
186-859	**12/1938**
187-858	**24/1938**
188-857	**24/1938**
189-856	**4/1938**
190-855	**12/1938**
191-854	**12/1938**
192-853	**24/1938**
193-852	**48/1938**
194-851	**48/1938**
195-850	**24/1938**
196-849	**12/1938**
197-848	**12/1938**
198-847	**24/1938**
199-846	**48/1938**
200-845	**48/1938**
201-844	**48/1938**
202-843	**6/1938**
203-842	**24/1938**
204-841	**6/1938**
205-840	**1/1938**
206-839	**24/1938**
207-838	**12/1938**
208-837	**48/1938**
209-836	**24/1938**
210-835	**12/1938**
211-834	**6/1938**
212-833	**12/1938**
213-832	**12/1938**
214-831	**24/1938**
215-830	**24/1938**
216-829	**24/1938**
217-828	**12/1938**
218-827	**24/1938**
219-826	**48/1938**
220-825	**24/1938**
221-824	**8/1938**
222-823	**48/1938**
223-822	**24/1938**
224-821	**24/1938**
225-820	**24/1938**
#226-819	**24/1938**
227-818	**48/1938**
228-817	**48/1938**
229-816	**12/1938**
230-815	**24/1938**
231-814	**48/1938**
232-813	**24/1938**
233-812	**24/1938**
234-811	**24/1938**
235-810	**6/1938**
236-809	**12/1938**
237-808	**2/1938**
238-807	**6/1938**
239-806	**12/1938**
240-805	**24/1938**
241-804	**24/1938**
242-803	**12/1938**
243-802	**12/1938**

e = 9 or 12

244-801	**6/8398**
245-800	**72/8398**
246-799	**72/8398**
247-798	**18/8398**
248-797	**72/8398**
249-796	**36/8398**
250-795	**3/8398**
251-794	**72/8398**
252-793	**24/8398**
253-792	**72/8398**
254-791	**12/8398**
255-790	**72/8398**
256-789	**36/8398**
257-788	**144/8398**
258-787	**18/8398**
259-786	**36/8398**
260-785	**72/8398**
261-784	**72/8398**
262-783	**72/8398**
263-782	**2/8398**
264-781	**12/8398**
265-780	**4/8398**
266-779	**36/8398**
267-778	**24/8398**
268-777	**12/8398**
269-776	**12/8398**
270-775	**36/8398**
271-774	**72/8398**
272-773	**144/8398**
273-772	**72/8398**

#274-771	**72/8398**	#325-720	**144/8398**	**e = 10 or 11**	
275-770	**18/8398**	326-719	**144/8398**		
276-769	**144/8398**	327-718	**72/8398**	#375-670	**1/16 796**
277-768	**24/8398**	328-717	**72/8398**	376-669	**60/16 796**
278-767	**36/8398**	329-716	**72/8398**	377-668	**40/16 796**
279-766	**36/8398**	330-715	**72/8398**	378-667	**30/16 796**
280-765	**18/8398**	331-714	**144/8398**	379-666	**120/16 796**
281-764	**72/8398**	332-713	**72/8398**	380-665	**30/16 796**
282-763	**144/8398**	333-712	**72/8398**	381-664	**120/16 796**
283-762	**144/8398**	334-711	**144/8398**	382-663	**120/16 796**
284-761	**36/8398**	335-710	**72/8398**	383-662	**120/16 796**
285-760	**72/8398**	336-709	**72/8398**	384-661	**60/16 796**
286-759	**144/8398**	337-708	**72/8398**	385-660	**120/16 796**
287-758	**36/8398**	338-707	**72/8398**	386-659	**24/16 796**
288-757	**36/8398**	339-706	**36/8398**	387-658	**15/16 796**
289-756	**9/8398**	340-705	**36/8398**	388-657	**120/16 796**
290-755	**72/8398**	341-704	**144/8398**	389-656	**20/16 796**
291-754	**12/8398**	342-703	**72/8398**	390-655	**60/16 796**
292-753	**18/8398**	343-702	**3/8398**	391-654	**60/16 796**
293-752	**72/8398**	344-701	**36/8398**	392-653	**60/16 796**
294-751	**72/8398**	345-700	**72/8398**	393-652	**240/16 796**
295-750	**36/8398**	346-699	**12/8398**	394-651	**60/16 796**
296-749	**36/8398**	347-698	**18/8398**	395-650	**15/16 796**
297-748	**24/8398**	348-697	**36/8398**	396-649	**120/16 796**
298-747	**72/8398**	349-696	**36/8398**	397-648	**60/16 796**
299-746	**6/8398**	350-695	**12/8398**	398-647	**120/16 796**
300-745	**72/8398**	351-694	**36/8398**	399-646	**10/16 796**
301-744	**72/8398**	352-693	**36/8398**	400-645	**30/16 796**
302-743	**36/8398**	353-692	**72/8398**	401-644	**120/16 796**
303-742	**72/8398**	354-691	**36/8398**	402-643	**120/16 796**
304-741	**144/8398**	355-690	**144/8398**	403-642	**60/16 796**
305-740	**72/8398**	356-689	**72/8398**	404-641	**10/16 796**
306-739	**72/8398**	357-688	**18/8398**	405-640	**120/16 796**
307-738	**72/8398**	358-687	**72/8398**	406-639	**120/16 796**
308-737	**144/8398**	359-686	**144/8398**	407-638	**20/16 796**
309-736	**36/8398**	360-685	**72/8398**	408-637	**120/16 796**
310-735	**144/8398**	361-684	**144/8398**	409-636	**120/16 796**
311-734	**144/8398**	362-683	**72/8398**	410-635	**120/16 796**
312-733	**144/8398**	363-682	**18/8398**	411-634	**20/16 796**
313-732	**72/8398**	364-681	**1/8398**	412-633	**240/16 796**
314-731	**72/8398**	365-680	**36/8398**	413-632	**240/16 796**
315-730	**72/8398**	366-679	**72/8398**	414-631	**120/16 796**
316-729	**72/8398**	367-678	**72/8398**	415-630	**120/16 796**
317-728	**12/8398**	368-677	**36/8398**	416-629	**240/16 796**
318-727	**72/8398**	369-676	**36/8398**	417-628	**60/16 796**
319-726	**36/8398**	370-675	**72/8398**	418-627	**120/16 796**
320-725	**144/8398**	371-674	**24/8398**	419-626	**120/16 796**
321-724	**24/8398**	372-673	**144/8398**	420-625	**240/16 796**
322-723	**144/8398**	373-672	**36/8398**	421-624	**240/16 796**
323-722	**144/8398**	374-671	**24/8398**	422-623	**120/16 796**
324-721	**144/8398**			423-622	**60/16 796**

v = 7

e = 10 or 11

#424-621	240/16 796
425-620	120/16 796
426-619	120/16 796
427-618	120/16 796
428-617	240/16 796
429-616	240/16 796
430-615	120/16 796
431-614	60/16 796
432-613	240/16 796
433-612	40/16 796
434-611	120/16 796
435-610	30/16 796
436-609	120/16 796
437-608	120/16 796
438-607	240/16 796
439-606	60/16 796
440-605	120/16 796
441-604	120/16 796
442-603	240/16 796
443-602	120/16 796
444-601	240/16 796
445-600	240/16 796
446-599	60/16 796
447-598	60/16 796
448-597	120/16 796
449-596	60/16 796
450-595	120/16 796
451-594	240/16 796
452-593	120/16 796
453-592	240/16 796
454-591	240/16 796
455-590	120/16 796
456-589	120/16 796
457-588	120/16 796
458-587	20/16 796
459-586	120/16 796
460-585	20/16 796
461-584	1/16 796
462-583	20/16 796
463-582	60/16 796
464-581	30/16 796
465-580	120/16 796
466-579	120/16 796
467-578	240/16 796
468-577	120/16 796
469-576	60/16 796
470-575	40/16 796
471-574	30/16 796
472-573	120/16 796
#473-572	60/16 796
474-571	60/16 796
475-570	20/16 796
476-569	240/16 796
477-568	240/16 796
478-567	60/16 796
479-566	60/16 796
480-565	120/16 796
481-564	120/16 796
482-563	120/16 796
483-562	240/16 796
484-561	240/16 796
485-560	120/16 796
486-559	60/16 796
487-558	240/16 796
488-557	60/16 796
489-556	120/16 796
490-555	120/16 796
491-554	30/16 796
492-553	120/16 796
493-552	60/16 796
494-551	120/16 796
495-550	240/16 796
496-549	120/16 796
497-548	120/16 796
498-547	240/16 796
499-546	60/16 796
500-545	60/16 796
501-544	30/16 796
502-543	20/16 796
503-542	240/16 796
504-541	120/16 796
505-540	30/16 796
506-539	120/16 796
507-538	60/16 796
508-537	240/16 796
509-536	240/16 796
510-535	240/16 796
511-534	120/16 796
512-533	240/16 796
513-532	60/16 796
514-531	120/16 796
515-530	120/16 796
516-529	120/16 796
517-528	60/16 796
518-527	120/16 796
519-526	120/16 796
520-525	30/16 796
521-524	60/16 796
522-523	60/16 796

The Rarest Graphs

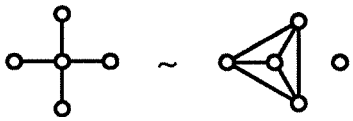
As we move through a poset of graphs, adding edges randomly, some graphs occur rarely, and beyond 5 vertices a few will likely never be built. Here are the rarest graphs, up to 8 vertices, with their probabilities.

v = 4

$\frac{1}{5}$

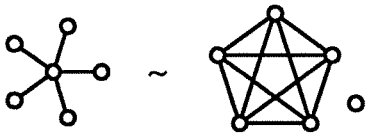
$\frac{1}{5}$

v = 5

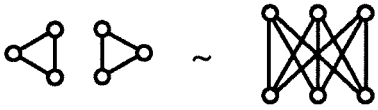


$\frac{1}{42}$

v = 6

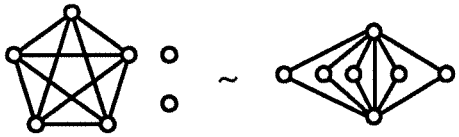


$\frac{2}{1001}$



$\frac{2}{1001}$

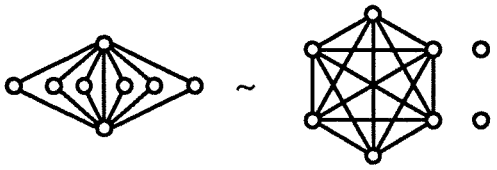
v = 7



$\frac{1}{16\,796}$

$\frac{1}{16\,796}$

v = 8



$\frac{1}{1\,337\,220}$