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Product Planning and Research
Ford Motor Company

One Parklane Boulevard
Dearborn, Michigan 48126

February 11, 1974

Dr. N. J. A. Sloane
Mathematics Research Center
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Murray Hill, N. J. 07974

Dear Dr. Sloane:

I have seen your very fine publication, "Handbook of Integer Sequences," on the new books shelf of the University of Michigan Dept. of Mathematics Library, and I marvel at the job you did in locating so many integer sequences, and then the proof-reading job was monumental! Of course, it included the Pell numbers sequence I was particularly interested in, and your book told me that it had that name, which I had not known before.

I ran across this sequence about a year and a half ago as a result of a computer program I wrote to generate right triangles with integer sides whose legs were consecutive integers. Then I recalled (never having had a course in number theory) having read once about primitive Pythagorean triples, so from my table of integer solutions I determined the values of relatively prime integers (Pell numbers) which basically determine right triangles of this kind. I had stumbled on to this situation through noticing the number 841 on a license plate was not only $29^2$ but also $20^2 + 21^2$, which gave me for the first time in my life an integer-sided right triangle besides 3,4,5 and 5,12,13. It is my observation that not very many people know 5,12,13 (let alone any other independent set) in addition to (3,4,5), which many people do know.

The way in which you indicated the derivation of a formula for the nth term surprised me, in coming from the fact that halves of the values of the 2nd differences reproduced the Pell numbers! Again, my lack of acquaintance with finite differences was the reason for my surprise.

How do you like this as an equivalent formula:

$$a_n = \frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}, \quad n = 1, \ldots, \infty?$$

I am a little bit brash to write $n = 1, \ldots, \infty$ as I do not know that it has been proved for all n. The formula is not mine, but was forwarded to me by a fellow Ford employee to whom I had shown my findings on integer-sided right triangles. He mentioned he had derived it "with some difficulty," and he did not include any steps, most of which he probably lost in his work at a relatively small blackboard as he frequently had to erase to obtain more working space. If you are interested in communicating with him, he is F. J. Mason, Transportation Research and Planning, Ford Motor Co., 10th floor, Village Plaza Tower, 23400 Michigan Avenue, Dearborn, MI, 48124. His phone number is (313) 323-1292, a Centrex system.

As a result of my personal excursion into the area of p.P.t.'s, besides the triangles generated by Pell numbers, I found out how to generate triangles whose longer leg and the hypotenuse were consecutive integers, and, of course, right triangles with neither of these restrictions. I felt it was suitable material for enrichment of a high school geometry course, and found that the Michigan Council of Teachers (high school) of Mathematics had an appropriate journal, MATHEMATICS IN MICHIGAN. My paper was published in it last May.

I suppose you see Prof. John Tukey from time to time. We met in the middle thirties at Brown University. Say hello to him for me, if you please.

Again, my sincere admiration for your very fine contribution to integer sequences.

Respectfully yours,

Calvin J. Kirchen

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February 28, 1974

Dr. Calvin J. Kirchen
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Dear Dr. Kirchen:

Thank you for your kind letter of February 11. I was interested to hear about how you discovered the Pell numbers. There is a nice elementary discussion of Pythagorean triangles and Pell equations in a Dover paperback, <u>Recreations in the Theory of Numbers - The Queen of Mathematics Entertains</u>, by A. H. Beiler, Chapters 14 & 22 - I think you would like this book.

Once you have the recurrence relation $a_n = 2a_{n-1} + a_{n-2}$ for the Pell numbers, it is really straightforward to deduce that

$$a_n = \frac{(1+\sqrt{2})^4 - (1-\sqrt{2})^n}{\sqrt{2}}, \quad n = 1,2,\ldots .$$

This is a standard calculation - see John Riordan's <u>Combinatorial Analysis</u>, page 27, equation (18). You write

$$a_n - 2a_{n-1} - a_{n-2} = 0$$

and find the roots of

$$x^2 - 2x - 1 = 0$$

which are $x = 1 + \sqrt{2}$ and $x = 1 - \sqrt{2}$. Then the solution is

$$a_n = A(1+\sqrt{2})^n + B(1-\sqrt{2})^n$$

and you determine $A = \frac{1}{2\sqrt{2}}$, $B = -\frac{1}{2\sqrt{2}}$ from the first values, $a_0 = 0$, $a_1 = 1$. (You could have asked a distinguished colleague of yours, Murray Klamkin, who is a world expert on solving mathematical problems - please give him my regards if you run into him.)

It was good to get your letter, and I am glad you liked my book.

With best regards,

MH-1216-NJAS-mv N. J. A. Sloane

P.S. Of course $(1-\sqrt{2})^n$ is very small if n is large, so

$a_n \doteqdot \frac{(1+\sqrt{2})^n}{2\sqrt{2}}$ with an error which is vanishingly small.