INTRODUCTION TO DE RHAM COHOMOLOGY

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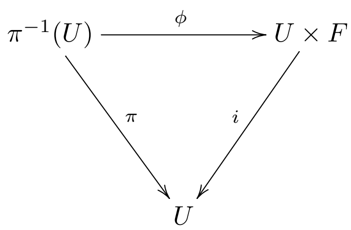
Abstract. We briefly review differential forms on manifolds. We prove homotopy invariance of cohomology, the Poincaré lemma and exactness of the Mayer–Vietoris sequence. We then compute the cohomology of some simple examples. Finally, we prove Poincaré duality for orientable manifolds.

Contents

1. Introduction

Definition 1.1. A fiber bundle E over a topological space B with fiber a given topological space F is a space along with a continuous surjection π such that for each $x \in E$ there exists a neighborhood $\pi(x) \in U \subset B$ and a homeomorphism ϕ such that the following diagram commutes,



where i is the usual map, $i(p, v) = p$. We call F the fiber and B the base space.

Example 1.2. The trivial bundle over a base space B with fiber F is just $B \times F$ with the maps $\pi = i$ and $\phi = id$.

Definition 1.3. A section $s \colon B \to E$ of a fiber bundle is a continuous map with the property that $\pi(s(p)) = p$ for all $p \in B$. If B and E are smooth manifolds, we require that s is smooth.

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Definition 1.4. A vector bundle is a fiber bundle where the fiber is a vector space and with the additional properties that:

(1) For all $x \in E$ and all corresponding $U, V \subset B$ and ϕ, ψ we have $\psi \circ \phi^{-1}(x, v) = (x, g_{UV}(v))$ for some invertible linear function g_{UV} depending only on U and V.

(2) For all U, V, W, as above, we have $g_{UU} = id$ and $g_{UV} \circ g_{VW} \circ g_{WU} = id$. Intuitively, any three transition maps must agree.

Remark 1.5. In this paper, we concern ourselves with vector bundles over manifolds.

Definition 1.6. A tangent vector v on a n-dimensional manifold M at a point p is an equivalence class of all smooth curves $\{\gamma_\alpha\}$ such that $\gamma_\alpha\colon (-1,1) \to M$, $\gamma_\alpha(0) = p$ and given some chart (U, f) such that $p \in f(U)$ we have

$$D(f^{-1} \circ \gamma_\alpha)\big|_{f^{-1}(p)} = D(f^{-1} \circ \gamma_\beta)\big|_{f^{-1}(p)},$$

for all $\gamma_\alpha \sim \gamma_\beta$ where $D(\cdot)\big|_{f^{-1}(p)}$ is just the usual derivative on Euclidean space at $f^{-1}(p)$. The space of all vectors at a point p is called the tangent space of M at p, denote T_pM. The space of all vectors at all points on the manifold is the tangent bundle TM and a section of the tangent bundle is a vector field on M.

Proposition 1.7. *For all M an n-dimensional manifold and $p \in M$ we have $T_pM \cong R^n$.*

Proposition 1.8. *The tangent bundle is a vector bundle.*

Definition 1.9. For a vector bundle E over the base space B with fiber F, we have the dual bundle E^* over base space B with fiber F^*,

$$\begin{aligned} E^* &\equiv \coprod_{p \in B} \pi^{-1}(p)^* \\ &\equiv \coprod_{p \in B} \{y\colon y \circ \phi^{-1} \text{ is a linear map on } \{p\} \times F\} \end{aligned}$$

We define $\pi^*(\pi^{-1}(p)) = \{p\}$ and $\phi^*(y) = y \circ \phi$. The new transition maps are given by $g^*_{UV} = (g^T_{UV})^{-1}$. The cotangent bundle is the dual of the tangent bundle, $(TM)^* \equiv T^*M$.

Definition 1.10. Define the k^{th} exterior power of a vector space F with itself

$$\Lambda^k F \equiv \bigotimes_k F/\sim,$$

where $v_1 \otimes \cdots \otimes w \otimes \cdots \otimes w \otimes \cdots \otimes v_k \sim 0$ and of course two elements of the exterior power are equivalent if their difference is 0. For the element $\{v_1 \otimes \cdots \otimes v_k\}$ we write $v_1 \wedge \cdots \wedge v_k$ and define $\wedge\colon \Lambda^k F \oplus \Lambda^m F \to \Lambda^{k+m} F$ by

$$\wedge(v_1 \wedge \cdots \wedge v_k, v_{k+1} \wedge \cdots \wedge v_{k+m}) \equiv v_1 \wedge \cdots \wedge v_{k+m}.$$

Definition 1.11. For a vector bundle E over the base space B with fiber F we define the k^{th} exterior power

$$\begin{aligned}\Lambda^k E &\equiv \coprod_{p\in B} \Lambda^k \pi^{-1}(p)\\ &\equiv \coprod_{p\in B} \{(x_1,\dots,x_k)\colon x_i \in E, \pi(x_i)=\pi(x_j)\forall i,j\}/\sim,\\ &\qquad \text{where } (x_1,\dots,x_k)\sim(y_1,\dots,y_k) \text{ whenever}\\ &\qquad \phi(x_1)\wedge\cdots\wedge\phi(x_k)=\phi(y_1)\wedge\cdots\wedge\phi(y_k).\end{aligned}$$

with maps $\Lambda^k\phi(x_1,\dots,x_k)=(p,\phi(x_1)\wedge\cdots\wedge\phi(x_k))$ and $\pi = i\circ\phi$. The transition maps $\Lambda^k g_{UV}$ are given by taking the k^{th} exterior power of the operator g_{UV} in the traditional, vector-space sense.

Remark 1.12. The dual and the exterior product not only define new *objects* in a *category* to associated objects. (We would say elements and set except the collection of all vectors spaces or all vector bundles is not properly a set since it's too big). Both the dual and the exterior product are *functors* because they also send *arrows* to arrows. (Again, we would say functions except we may want to look at certain kinds of functions like strictly linear functions or may want to use objects which aren't sets). This is true on bundles as well as on vector spaces. Given a linear function $f\colon V\to W$, we get a function $f^*\colon W^*\to V^*$ where $f^*(v)(w)=v^*(f(w))$. Similarly, given a linear function $f\colon V\to W$, we get another function $\Lambda^k f\colon \Lambda^k V\to \Lambda^k W$ by $f(v_1\wedge\cdots\wedge v_k)=f(v_1)\wedge\cdots\wedge f(v_k)$.

Definition 1.13. The space of k-forms on a manifold M, denoted $\Omega^k(M)$ or Ω^k when the manifold is clear, is the space of sections of $\Lambda^k T^*M$.

Definition 1.14. Pick a basis $v_1,\dots,v_n$ for a fiber R at a point $p\in M$. Define the exterior derivative $d\colon \Omega^k\to\Omega^{k+1}$ at p by

$$d(a(p)w_1\wedge\cdots\wedge w_k)\equiv \sum_{1\le m\le n}\frac{\partial a}{\partial v_m}\Big|_p v_m\wedge w_1\wedge\cdots\wedge w_k.$$

This extends linearly to sums.

Definition 1.15. A chain complex is a sequence of abelian groups $C_0,C_1,C_2,\dots$ and a sequence of homomorphism called boundary maps

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \xleftarrow{\partial_3} \cdots$$

with the property that $\partial_k\circ\partial_{k+1}=0$ for all k. Then the k^{th} homology group is defined by $H_k\equiv \ker(\partial_k)/\mathrm{im}(\partial_{k+1})$. Notice that this is well defined since the groups are abelian and $\partial_k\circ\partial_{k+1}=0$ implies $\ker(\partial_k)\supset\mathrm{im}(\partial_{k+1})$.

A cochain complex is a sequence of abelian groups $C_0,C_1,C_2,\dots$ and a sequence of homomorphisms called coboundary maps

$$C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \cdots$$

with the property that $\delta^{k+1}\circ\delta^k=0$. Then the k^{th} cohomology group is defined by $H^k\equiv\ker(\delta^k)/\mathrm{im}(\delta^{k-1})$.

Lemma 1.16. *The exterior derivative is a homomorphism with the property that* $d^2=0$.

Definition 1.17. The De Rham cohomology of a manifold is the cohomology generated by the cochain complex

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \cdots$$

If $\omega \in \ker(d)$ we say ω is closed. If $\omega \in \mathrm{im}(d)$ we say ω is exact. For the rest of the paper, we deal with De Rham cohomology.

2. Smooth Homotopy Invariance

Definition 2.1. Given a map $f\colon M \to N$, the pullback map $f^*\colon \Omega^k(N) \to \Omega^k(M)$ is defined by

$$f^*(\omega)\big|_p(v_1, \ldots, v_k) \equiv \omega\big|_{f(p)}(Df^{-1}v_1, \ldots, Df^{-1}v_k).$$

Proposition 2.2. *The pullback commutes with the exterior derivative i.e. the following diagram commutes.*

$$\begin{array}{ccc} \Omega^k(N) & \xrightarrow{\quad d \quad} & \Omega^{k+1}(N) \\ \big\downarrow f^* & & \big\downarrow f^* \\ \Omega^k(M) & \xrightarrow{\quad d \quad} & \Omega^{k+1}(M) \end{array}$$

Proposition 2.3. *The pullback commutes with the wedge product i.e. the following diagram commutes.*

$$\begin{array}{ccc} \Omega^k(N) \oplus \Omega^m(N) & \xrightarrow{\quad \wedge \quad} & \Omega^{k+m}(N) \\ \big\downarrow f^* & & \big\downarrow f^* \\ \Omega^k(M) \oplus \Omega^m(M) & \xrightarrow{\quad \wedge \quad} & \Omega^{k+m}(M) \end{array}$$

Definition 2.4. We say a map $h\colon M \times I \to N$ is a smooth homotopy if it is smooth. We say two maps are smoothly homotopic, $f\colon M \to N$ and $g\colon M \to N$, if there exists a smooth homotopy h with $h(x, 0) = f(x)$ and $h(x, 1) = g(x)$. We say two manifolds M and N are smooth homotopy equivalent if there exist smooth maps $f\colon M \to N$ and $g\colon N \to M$ such that $g \circ f$ and $f \circ g$ are smoothly homotopic to the identity maps on M and N.

Theorem 2.5. *If two manifolds M and N are smoothly homotopy equivalent then their k^{th} cohomology groups are isomorphic for all k.*

Proof. Suppose $f\colon M \to N$ and $g\colon N \to M$ as above, with $h\colon M \times I \to M$ such that $h(x, 0) = x$ and $h(x, 1) = g(f(x))$. Let $\omega \in H^k((g \circ f)(M))$. Then there exists a pullback

$$h^*(\omega) \equiv \eta + dt \wedge \alpha,$$

where $\eta \in H^k(M \times I)$, $\alpha \in H^{k-1}(M \times I)$, neither η nor α contain a dt term and t is the coordinate in I. Furthermore, for all $t_0 \in I$, $h(\cdot, t_0)\colon M \to M$ induces the pullback

$$h^*(\cdot, t_0)(\omega) = \eta\big|_{t=t_0}$$

Since η varies smoothly in time, we can apply the fundamental theorem of calculus,

$$h^*(\cdot, 1)(\omega) - h^*(\cdot, 0)(\omega) = \eta\big|_{t=1} - \eta\big|_{t=0} = \int_0^1 \frac{\partial \eta}{\partial t} dt$$

We have two exterior derivative operators which we will temporarily distinguish, d_M and $d_{M\times I}$. Because ω is closed and the pullback commutes with the exterior derivative,

$$0 = h^*(d_M(\omega)) = d_{M\times I}(h^*(\omega)) = dt \wedge (\frac{\partial \eta}{\partial t} - d_M(\alpha)).$$

So $\frac{\partial \eta}{\partial t} = d_M(\alpha)$. Combining these last two equations yields,

$$h^*(\cdot, 1)(\omega) - h^*(\cdot, 0)(\omega) = \int_0^1 d_M(\alpha) dt = d_M \left(\int_0^1 \alpha dt \right)$$

which is an exact form. Therefore, $(g \circ f)^* = h^*(\cdot, 1)$ is the identity map on H^k so f^* and g^* are inverses so they are isomorphisms.

□

Corollary 2.6 (Poincaré Lemma)**.** *Every closed form is locally exact.*

Proof. Take a ball around any point p. Every ball is contractible. We just showed that $H^k(R^{n-1}) \cong H^k(R^{n-1} \times I)$ so we only need to show that $H^n(R^n) = 0$, for instance by explicit construction: $d(\int_p^{x_0} f dx_n \cdot x_1 \wedge \cdots \wedge x_{n-1}) = f \cdot x_1 \wedge \cdots \wedge x_n$ Then apply theorem 2.2. □

Definition 2.7. Define the Euler Characteristic on a closed, oriented manifold M of dimension n by

$$\chi(M) \equiv \sum_k (-1)^k \dim H^k(M)$$

Corollary 2.8. *If two manifolds M and N are smoothly homotopy equivalent, then $\chi(M) = \chi(N)$.*

Proof. Since the dimension of each cohomology class is homotopy invariant, so is any function of the dimension. □

Remark 2.9. The Euler Characteristic can also be easily stated in the language of homology, where it agrees with the usual notion of triangulating a two-dimensional surface.

3. Mayer–Vietoris

Definition 3.1. For the rest of the paper, U and V are open sets whose union forms a manifold. Define the restriction map *res*: $H^k(U \cup V) \to H^k(U) \oplus H^k(V)$ by *res*$(\omega) \equiv (\omega\big|_U, \omega\big|_V)$.

Definition 3.2. Define the difference map *diff*: $H^k(U) \oplus H^k(V) \to H^k(U \cap V)$ by *diff*$(\omega, \eta) \equiv \omega - \eta$.

Definition 3.3. Define the coboundary map *cobd*: $H^k(U \cap V) \to H^{k+1}(U \cup V)$ by

$$cobd(\omega)_p \equiv \begin{cases} d(\alpha)_p & : p \in U \\ d(\beta)_p & : p \in V \end{cases}$$

where α is a k-form on U, β is a k-form on V, and $\alpha - \beta = \omega$ on $U \cap V$. Since ω is closed, $d\omega = 0 = d\alpha - d\beta$ on $U \cap V$. Notice, this map is obtained by extending ω to the whole manifold, then applying the coboundary map.

Proposition 3.4. *The coboundary map is well defined up to exact forms.*

Definition 3.5. We say a sequence $X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \cdots$ is exact if $\mathrm{im}(f_n) = \ker(f_{n+1})$ for all n.

Theorem 3.6. *The sequence $H^k(U \cup V) \xrightarrow{res} H^k(U) \oplus H^k(V) \xrightarrow{diff} H^k(U \cap V)$ is exact.*

Proof. Let $\omega \in H^k(U \cup V)$. Then $res(\omega) = (\omega|_U, \omega|_V)$ and therefore, $diff(res(\omega)) = diff(\omega|_U, \omega|_V) = 0$. Thus, $\mathrm{im}(res) \subset \ker(diff)$.

Let $(\omega, \eta) \in \ker(diff)$, so $\omega = \eta$ on $U \cap V$. Then define

$$\alpha_p = \begin{cases} \omega_p & : p \in U \\ \eta_p & : p \in V \end{cases}$$

Then $res(\alpha) = (\omega, \eta)$ so $\mathrm{im}(res) \supset \ker(diff)$. □

Theorem 3.7. *The sequence $H^k(U) \oplus H^k(V) \xrightarrow{diff} H^k(U \cap V) \xrightarrow{cobd} H^{k+1}(U \cup V)$ is exact.*

Proof. Let $(\omega, \eta) \in H^k(U) \oplus H^k(V)$. Then $cobd(diff(\omega, \eta)) = cobd(\omega - \eta)$. We can choose $\alpha = \omega$, $\beta = \eta$ with α, β as in definition 3.3. Therefore,

$$cobd(\omega)_p = \begin{cases} d(\omega)_p & : p \in U \\ d(\eta)_p & : p \in V \end{cases}$$

But since ω and η are closed, $cobd(\omega) = 0$. Thus, $\mathrm{im}(diff) \subset \ker(cobd)$.

Next, let $\omega \in \ker(cobd)$ and find α, β as in defintion 3.3. Since $\omega \in ker(cobd)$, $d\alpha = 0$ and $d\beta = 0$. Therefore, $(\alpha, \beta) \in H^k(U) \oplus H^k(V)$. Then $diff(\alpha, \beta) = \alpha - \beta = \omega$ by their defining property in definition 3.3. Thus, $\mathrm{im}(diff) \supset \ker(cobd)$. □

Theorem 3.8. *The sequence $H^k(U \cap V) \xrightarrow{cobd} H^{k+1}(U \cup V) \xrightarrow{res} H^{k+1}(U) \oplus H^{k+1}(V)$ is exact.*

Proof. Let $\omega \in U \cap V$, α, β as in definition 3.3. Then $res(cobd(\omega)) = (d\alpha, d\beta)$ which is exact by definition. Thus, $\mathrm{im}(cobd) \subset \ker(res)$.

Finally, let $\omega \in \ker(res)$. Then $\omega = d(\alpha)$ on U for some α and $\omega = d(\beta)$ on V for some β. Then $\alpha - \beta \in H^k(U \cap V)$. By definition,

$$cobd(\alpha - \beta)_p = \begin{cases} d(\alpha)_p & : p \in U \\ d(\beta)_p & : p \in V \end{cases}$$

But $d(\alpha) = \omega|_U$ and $d(\beta) = \omega|_V$ so $\mathrm{im}(cobd) \supset \ker(res)$. □

Definition 3.9. The Mayer–Vietoris sequence is the sequence

$$\begin{aligned} 0 \to H^0(U \cup V) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \\ \to H^1(U \cup V) \to H^1(U) \oplus H^0(V) \to H^1(U \cap V) \cdots \end{aligned}$$

Theorem 3.10. *The Mayer–Vietoris sequence is exact.*

4. Applications

Example 4.1. The cohomology of a ball is trivial. In particular, $\dim H^k(R^n) = \delta(k,0)$.

Proof. Just apply the Poincaré lemma for k-forms where $k \neq 0$. When $k = 0$, notice that constant functions are closed. □

Example 4.2. If S^n is the n-sphere, $\dim H^k(S^n) = \delta(n,k)$.

Proof. We proceed by induction on n and leave the base case to the reader. Let $U = \{(x,y,z,\ldots) \in S^n : x < \frac{1}{2}\}$ and $V = \{(x,y,z,\ldots) \in S^n : x > -\frac{1}{2}\}$ so the intersection $U \cap V$ is smoothly homotopy equivalent to the S^{n-1} sphere and U, V are smoothly homotopy equivalent to the $n-1$ ball. Then $\cdots \to 0 \to H^{k-1}(S^{n-1}) \to H^k(S^n) \to 0 \to \cdots$ is exact . By induction, $\dim H^k(S^n) = \delta(n,k)$. □

Example 4.3. If $T^n = (S^1)^n$, then $\dim H^k(T^n) = \binom{n}{k}$.

Proof. Again we use induction on n. Let $U = \{(\theta_1,\theta_2,\theta_3,\ldots) \in T^n : \theta_1 \in (\frac{1}{6},\frac{5}{6})\}$ and $V = \{(\theta_1,\theta_2,\theta_3,\ldots) \in T^n : \theta_1 \in (\frac{4}{6},\frac{2}{6})\}$ so the intersection $U \cap V$ is smoothly homotopy equivalent to $T^{n-1} \coprod T^{n-1}$ and U, V are smoothly homotopy equivalent to T^{n-1}. The base case is covered by S^1 in example 4.2. Notice that $H^k(U) \oplus H^k(V) \xrightarrow{diff} H^k(U \cap V)$ induces a map $H^k(U) \times \{0\} \to H^k(U \cap V)$ injectively. Furthermore, by symmetry we can take the opposite map on V to get that $\dim \ker(\mathit{diff}) = \dim\ \mathrm{im}(\mathit{diff}) = \binom{n-1}{k}$ if $H^k(U) \oplus H^k(V) \xrightarrow{diff} H^k(U \cap V)$. Thus,

$$H^{k-1}(T^{n-1})^2 \cong R^{2\binom{n-1}{k-1}} \xrightarrow[\dim \ker = \binom{n-1}{k-1}]{\dim \mathrm{im} = \binom{n-1}{k-1}} H^{k-1}(T^{n-1})^2 \cong R^{2\binom{n-1}{k-1}} \xrightarrow[\dim \ker = \binom{n-1}{k-1}]{\dim \mathrm{im} = \binom{n-1}{k-1}}$$

$$H^k(T^n) \xrightarrow[\dim \ker = \binom{n-1}{k-1}]{\dim \mathrm{im} = \binom{n-1}{k}} H^k(T^{n-1})^2 \cong R^{2\binom{n-1}{k}} \xrightarrow[\dim \ker = \binom{n-1}{k}]{\dim \mathrm{im} = \binom{n-1}{k}} H^k(T^{n-1})^2,$$

so $\dim H^k(T^n) = \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$. □

5. Poincaré Duality

Definition 5.1. We denote by $\Omega^k_C(M) = \Omega^k_C$ the space of k-forms with compact support. The exterior derivative induces a map on this subspace. Then the cochain complex

$$\Omega^0_C \xrightarrow{d} \Omega^1_C \xrightarrow{d} \Omega^2_C \xrightarrow{d} \cdots$$

gives rise to the compactly supported cohomology, denoted $H^k_C(M)$.

Next, we state a simple but helpful lemma.

Lemma 5.2 (Five Lemma)**.** *Suppose, in the following commutative diagram, that each set is an abelian group, that each map is a homomorphism, that the rows form* *exact sequences and that j, k, m and n are isomorphisms.*

$$\begin{array}{ccccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{i} & E \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle n} \\
A' & \xrightarrow{o} & B' & \xrightarrow{p} & C' & \xrightarrow{q} & D' & \xrightarrow{r} & E'
\end{array}$$

Then l is an isomorphism.

Proof. First, suppose $c \in \ker(l)$. Then $m(h(c)) = q(l(c)) = 0$ because the diagram commutes. Because m is injective, we have $h(c) = 0$. Since the top row is exact, $\ker(h) = \mathrm{im}(g)$ so there exists $b \in B$ such that $g(b) = c$. Again, the diagram commutes so $l(g(b)) = p(k(b)) = 0$. Since the bottom row is exact, $\ker(p) = \mathrm{im}(o)$ so there exists $a' \in A'$ such that $o(a') = k(b)$. Because j is surjective, there exists $a \in A$ such that $j(a) = a'$. Then $o(j(a)) = k(f(a)) = k(b)$ but because k is an isomorphism, $f(a) = b$. Then $0 = g(f(a)) = g(b) = c = 0$ so l is injective.

Second, let $c' \in C'$. Then because m is surjective, there exists $d \in D$ such that $q(c') = m(d)$. Thus, $r(q(c')) = r(m(d)) = n(i(d))$ using commutativity. Because the bottom row is exact, $\mathrm{im}(q) = \ker(r)$ and therefore $r(q(c')) = 0 = n(i(d))$ applying the previous line. Since n is injective, $i(d) = 0$. Because the top row is exact, $\ker(i) = \mathrm{im}(h)$ and therefore there exists $c \in C$ such that $h(c) = d$. By linearity and the defining property of d we have, $q(c' - l(c)) = q(c') - q(l(c)) = m(d) - m(h(c)) = 0$. Because the bottom row is exact $\ker(q) = \mathrm{im}(p)$ so there exists $b' \in B'$ such that $p(b') = c' - l(c)$. Since k is surjective, there exists $b \in B$ such that $k(b) = b'$. Since the diagram commutes, $l(g(b)) = p(k(b)) = c' - l(c)$ so $l(g(b) + c) = c'$ proving l is surjective. □

We define a map $\int\colon H^k(M) \oplus H^{n-k}_C(M) \to R$ by $\int(\omega, \eta) = \int_M \omega \wedge \eta$. Since η is compactly supported, $\int(\omega, \eta)$ is finite. Clearly, $\int(\omega, \cdot) \equiv I_M(\omega)(\cdot)$ defines a linear map $I_M\colon H^k(M) \to (H^{n-k}_C)^*$. Although the map I_M implicitly depends on k, we suppress this dependence for the sake of cleanliness. The main theorem of this section states that for a broad class of spaces, I_M is an isomorphism so, in particular, $H^k(M) \cong H^{n-K}_C(M)$ and $\dim H^k(M) = \dim H^{n-k}_C(M)$.

Lemma 5.3. *If $U \cup V$ is orientable and if I_U, I_V and $I_{U\cap V}$ are isomorphisms for all k, so is $I_{U\cup V}$.*

Proof. We use Mayer–Vietoris and the five lemma. We will check that the following diagram commutes up to sign with an appropriate choice of maps.

$$\begin{array}{ccccccc}
\longrightarrow & H^{k-1}(U\cap V) & \longrightarrow & H^k(U\cup V) & \longrightarrow & H^k(U) \oplus H^k(V) & \longrightarrow \\
 & \downarrow{\scriptstyle I_{U\cap V}} & & \downarrow{\scriptstyle I_{U\cup V}} & & \downarrow{\scriptstyle I_U \oplus I_V} & \\
\longrightarrow & H^{n-k+1}_C(U\cap V)^* & \longrightarrow & H^{n-k}_C(U\cup V)^* & \longrightarrow & H^{n-k}(U)^*_C \oplus H^k(V)^*_C & \longrightarrow
\end{array}$$

The top line is the Mayer–Vietoris sequence. Define *sum*: $H^k_C(U) \oplus H^k_C(V) \to H^k_C(U \cup V)$ by $sum(\omega, \eta) \equiv \omega + \eta$. Define the inclusion map *inc*: $H^k_C(U \cap V) \to$

$H^k_C(U) \oplus H^k_C(V)$ by $inc(\omega) \equiv (\omega\big|_U, -\omega\big|_V)$. We can take the dual to get a sequence in the opposite direction. The proof that this sequence is exact is similar to the proof of Mayer–Vietoris and is omitted.

First, if f, g, η and α are defined as in definition 3.3 and $\beta \in H^{n-k}_C(U \cup V)$ then, recalling that $df = 0$ outside of $U \cap V$,

$$I_{U\cup V}(cobd(\omega))(\beta) = \int_{U\cup V} d(f\eta - g\alpha) \wedge \beta = \int_{U\cap V} df\omega \wedge \beta = \pm d^*(I_{U\cap V}(\omega))(\beta).$$

Second, if $(\alpha, \beta) \in H^{n-k}_C(U) \oplus H^{n-k}_C(V)$ and $\omega \in H^k(U \cup V)$ then,

$$\begin{aligned} I_U \oplus I_V(res(\omega))(\alpha, \beta) &= I_U \oplus I_V(\omega\big|_U, \omega\big|_V)(\alpha, \beta) = \int_U \omega \wedge \alpha + \int_V \omega \wedge \beta \\ &= \int_{U\cup V} \omega \wedge (\alpha + \beta) = sum^*(I_{U\cup V}(\omega))(\alpha, \beta), \end{aligned}$$

again using that α, β only have support on U and V respectively. Lastly, if $(\omega, \eta) \in H^k(U) \oplus H^k(V)$ and $\alpha \in H^{n-k}_C(U \cap V)$ then

$$I_{U\cap V}(diff(\omega, \eta))(\alpha) = \int_{U\cap V} (\omega - \eta) \wedge \alpha = inc^*(I_U \oplus I_V(\omega, \eta))(\alpha, \beta).$$

By the five lemma, $I_{U\cup V}$ is an isomorphism. □

Definition 5.4. An open cover $\{U_\alpha\}$ of a manifold M is a good cover if every finite intersection $U_1 \cap \cdots \cap U_m$ is contractible.

Theorem 5.5 (Poincaré Duality)**.** *If a manifold M is orientable and has a finite good cover, then I_M is an isomorphism.*

Proof. We proceed by induction. When $M = R^n$, then $\dim H^k(R^n) = \delta(0, k)$. Notice that constant functions are not in $\Omega^0_C(R^n)$ so $\dim H^0_C(R^n) = 0$. To check that $\dim H^n_C(R^n) = 1$, we notice that $\int : \Omega^k \to R$ is nonzero but $\int_K d(\alpha) = \int_{\partial K} \alpha = 0$ by Stokes. Furthermore, $I_{R^n}(1)(\alpha) = \int_{R^n} \alpha$ so $I_{R^n}(1) = id \neq 0$ which proves that the map is non-degenerate and so an isomorphism.

Next, if M has a finite good cover, we proceed by induction on the size of the good cover using lemma 5.3. Namely, if Poincaré duality holds for any manifold covered by a good cover of size $m-1$ and if $\{U_1, \ldots U_m\}$ is a good cover of M, then Poincaré duality holds for $U_m \cong R^n$, it holds for $U_1 \cup \cdots \cup U_{m-1}$ by assumption and it holds for $\cap U_i \cong R^n$. By lemma 5.3, it holds for M. □

Remark 5.6. The theorem holds even without assuming a *finite* good cover, but it is most easily and appropriately proven with more advanced technology. One can show that any (Hausdorff, second-countable) manifold has a good cover by taking a neighborhood of each point which is convex in some chart, then noting that the intersection of two convex subsets of R^n is convex and so contractible by the multiplication map. By shrinking the size of each neighborhood, we can ensure that any other element of the good cover lies in a compatible chart.[1]

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[1]This proof that every manifold has a good cover comes from [6]. For the most accessible proof that Poincaré duality for finite good covers implies Poincaré duality for infinite good covers comes from [7] pages 197-200. More advanced proofs involving a slightly different formulation can be found in, for instance, [8].

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References

[1] Ben Andrews. Lectures on Differential Geometry. http://maths-people.anu.edu.au/ andrews/DG/DG_chap15.pdf
[2] Raoul Bott and Loring Tu. Differential Forms in Algebraic Topology. Springer. 1982.
[3] Brian Conrad. Differential Geometry Handouts. http://math.stanford.edu/ conrad/diffgeomPage/handouts/bundleops.pdf
[4] Fiber Bundle. http://en.wikipedia.org/wiki/Fiber_bundle
[5] Daniel S. Freed. Fiber Bundles and Vector Bundles. https://www.ma.utexas.edu/users/dafr/M392C/Notes/FiberBundles.pdf
[6] Proving the Existence of Good Covers. http://mathoverflow.net/questions/102161/proving-the-existence-of-good-covers
[7] Werner Greub, Stephen Halperin and Ray Vanstone. Connections, Curvature and Cohomology. Academic Press. 1972.
[8] Hikaru Kiyo. Poincaré Duality. http://www.math.uchicago.edu/ may/VIGRE/VIGRE2009/ REUPapers/Kiyo.pdf
[9] Tony Perkins. Vector Bundles, Connections and Curvature. http://toperkin.mysite.syr.edu/talks/vector_bundles_connections_curvature.pdf
[10] Aleskey Zinger. Notes on Vector Bundles. http://www.math.sunysb.edu/ azinger/mat531-spr10/vectorbundles.pdf