18.745 Introduction to Lie Algebras October 22, 2010

Lecture 14 — The Structure of Semisimple Lie Algebras III

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In this lecture we will prove more consequences of semisimplicity. Then we will prove a theorem that is used to prove that a Lie algebra is semisimple. Lastly, we will begin to examine examples of semisimple Lie algebras.

Recall some of the facts that we already know about semisimple Lie algebras. Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{F}$, an algebraically closed field of characteristic 0. Choose a Cartan subalgebra $\mathfrak{h}$ and consider the root space decomposition:

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right), \quad [\mathfrak{h}, \mathfrak{h}] = 0, \quad dim \mathfrak{g}_\alpha = 1.$$

We use this decomposition to rewrite the Killing form K as a sum over root spaces. For $h_1, h_2 \in \mathfrak{h}$ we have:

$$K(h_1, h_2) = tr_{\mathfrak{g}}(\mathbf{ad}\ h_1)(\mathbf{ad}\ h_2) = \sum_{\alpha \in \Delta} \alpha(h_1)\alpha(h_2). \tag{1}$$

We proved in lecture 12 that $K|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate and creates an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ by $\nu(h)(h') = K(h, h')$. From this isomorphism we define a bilinear form K on $\mathfrak{h}^*$:

$$K(\lambda_1, \lambda_2) = \sum_{\alpha \in \Delta} \alpha(\nu^{-1}(\lambda_1))\alpha(\nu^{-1}(\lambda_2)) = \sum_{\alpha \in \Delta} K(\lambda_1, \alpha)K(\lambda_2, \alpha). \tag{2}$$

Definition 14.1. $\mathfrak{h}^*_{\mathbb{Q}} \subset \mathfrak{h}^*$ is the $\mathbb{Q}$-*span of* Δ. (Note that $\mathbb{Q} \subset \mathbb{F}$ since $charF = 0$.)

Now, a few consequences of semisimplicity:

Theorem 14.1. *For* $\mathfrak{g}$ *as above, the following are true:*

1. Δ *spans* $\mathfrak{h}^*$ *over* $\mathbb{F}$.
2. $K(\alpha, \beta) \in \mathbb{Q}$ *for all* $\alpha, \beta \in \Delta$.
3. $K|_{h^*_{\mathbb{Q}}\times\mathfrak{h}^*_{\mathbb{Q}}}$ *is a positive definite symmetric bilinear form with values in* $\mathbb{Q}$.

Proof. 1. Suppose Δ does not span $\mathfrak{h}^*$, then there exists a non-zero $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Delta$. This implies that $[h, \mathfrak{g}_\alpha] = 0$ for all α from the definition of root space. It is also true that $[h, \mathfrak{h}] = 0$, as from lecture 12 the Cartan subalgebra $\mathfrak{h}$ is abelian. Therefore, h is in $Z(\mathfrak{g})$. But $\mathfrak{g}$ is semisimple, so $Z(\mathfrak{g}) = 0$, which is a contradiction.

2. From Equation (2) we have:

$$K(\lambda, \lambda) = \sum_{\alpha \in \Delta} K(\lambda, \alpha)^2 \quad \forall \lambda \in \mathfrak{h}^*. \tag{3}$$

For $\lambda \in \Delta$, $K(\lambda, \lambda) \neq 0$, so:

$$\frac{4}{K(\lambda, \lambda)} = \sum_{\alpha \in \Delta} \left(\frac{2K(\lambda, \alpha)}{K(\lambda, \lambda)} \right)^2, \quad \lambda \in \Delta. \tag{4}$$

It follows that $\frac{4}{K(\lambda,\lambda)} \in \mathbb{Z}$ for $\lambda \in \Delta$ because by the string condition $\frac{2K(\lambda,\alpha)}{K(\lambda,\lambda)} = p - q \in \mathbb{Z}$. This implies that $K(\lambda, \lambda) \in \mathbb{Q}$ for $\lambda \in \Delta$.

But since $\frac{2K(\alpha,\beta)}{K(\alpha,\alpha)} \in \mathbb{Z}$ for $\alpha, \beta \in \Delta$ and $K(\alpha, \alpha) \in \mathbb{Q}$ we conclude that $K(\alpha, \beta) \in \mathbb{Q}$.

3. From part 2, $K(\alpha, \beta) \in \mathbb{Q}$, hence $K(\lambda, \alpha) \in \mathbb{Q}$ for $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ $\alpha \in \Delta$. Since by part 1, Δ spans $\mathfrak{h}^*$ and the Killing form is non-degenerate on $\mathfrak{h}$, its restriction to $\mathfrak{h}^*_{\mathbb{Q}}$ is non-degenerate as well.

Hence by equation (3), $K(\lambda, \lambda) \geq 0$ for $\lambda \in \mathfrak{h}^*_{\mathbb{Q}}$ since it is the sum of rational squares. This proves that K is positive and semi-definite.

It is a theorem of linear algebra that any non-degenerate positive semi-definite symmetric bilinear form is positive definite. This proves part 3. □

Every semisimple algebra is the direct sum of simple algebras. The following exercise shows that this decomposition is unique up to permutation.

Exercise 14.1. Recall that a semisimple Lie algebra $\mathfrak{g} = \bigoplus_{j=1}^N s_j$ where s_j are simple Lie algebras.

Prove that this decomposition is unique up to permutation of the summands and prove that any ideal of $\mathfrak{g}$ is a subsum of this sum.

Proof. We first prove the second part, take any ideal $\mathfrak{h}$. $\mathfrak{h} \cap s_j$ is an ideal of the subalgebra s_j. s_j is simple so this is either 0 or s_j. So $\mathfrak{h} = \bigoplus_{j=1}^N (\mathfrak{h} \cap s_j) = \bigoplus_{j \in A} s_j$, which proves the second part.

For the first part, consider a second decomposition $\mathfrak{g} = \bigoplus_{i=1}^M (t_i)$. Each t_i is an ideal and therefore is the direct sum of a subset of the s_j. t_i is simple, so it must be the direct sum of exactly one s_j. Similarly, each s_j is the direct sum of exactly one t_i. This proves the first part. □

We will now examine how the decomposition of a Lie algebra corresponds to a decomposition of its root space.

Let $\mathfrak{g}$ be the direct sum of two ideals, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where both $\mathfrak{g}_i$ are semisimple. Consider for each of them the root space decomposition. To do this we choose a Cartan subalgebra $\mathfrak{h}_i$ in each $\mathfrak{g}_i$.

$$\mathfrak{g}_i = \mathfrak{h}_i \oplus \left(\bigoplus_{\alpha \in \Delta_i} \mathfrak{g}_\alpha \right) \qquad \mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \right)$$

Where $\mathfrak{h}$ is $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ and $\Delta = \Delta_1 \sqcup \Delta_2$.

If $\alpha \in \Delta_1$ and $\beta \in \Delta_2$ and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$, then we have that $\alpha + \beta \notin \Delta$ and $\alpha + \beta \neq 0$. In other words,

$$\Delta = \Delta_1 \sqcup \Delta_2, \qquad \alpha + \beta \notin \Delta \cup 0 \ if \ \ \alpha \in \Delta_1, \beta \in \Delta_2. \tag{5}$$

Definition 14.2. The set Δ in a vector space V is called *indecomposable* if it cannot be decomposed into a disjoint union of non-empty subsets Δ_1 and Δ_2 such that equation 5 holds.

With this notion, we can give a simple way to check if a semisimple Lie algebra is simple.

Theorem 14.2. *Let $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta} g_\alpha)$ be a decomposition of a finite dimensional Lie algebra $\mathfrak{g}$ into a direct sum of subspaces such that the following properties hold:*

1. *$\mathfrak{h}$ is an abelian subalgebra and $dim g_\alpha = 1$ for all $\alpha \in \Delta, where \mathfrak{g}_\alpha = \{a \in \mathfrak{g} | [h, a] = \alpha(h)a \forall h \in \mathfrak{h}\}$,*
2. *$[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{F}h_\alpha$, where $h_\alpha \in \mathfrak{h}$ is such that $\alpha(h) \neq 0$,*
3. *$\mathfrak{h}^*$ is spanned by Δ.*

Then $\mathfrak{g}$ is a semisimple Lie algebra. Moreover if the set Δ is indecomposable, then $\mathfrak{g}$ is simple.

Lemma 14.3. *Let $\mathfrak{h}$ be an abelian Lie algebra and π its representation in a vector space V such that V has a weight space decomposition: $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$, where $V_\lambda = \{v \in V | \pi(h)v = \lambda(h)v, \forall h \in \mathfrak{h}\}$.*

If $U \subset V$ is a $\pi(\mathfrak{h})$-invariant subspace, then $U = \bigoplus_{\lambda\in\mathfrak{h}^*}(U \cap V_\lambda)$.

Proof. For $u \in U$, let:

$$u = \sum_{i=1}^{n} v_{\lambda_i}, \qquad v_{\lambda_i} \in V_{\lambda_i}, \qquad \lambda_i \neq \lambda_j. \tag{6}$$

We will prove that all v_{λ_i} are in U by induction on n. For the case $n = 1$, $v_{\lambda_1} = u \in U$.

For the case, $n > 1$ we apply $\pi(h)$ to both sides:

$$\pi(h)u = \sum_{i=1}^{n} \lambda_i(h) v_{\lambda_i} \tag{7}$$

where $h \in \mathfrak{h}$ is chosen such that $\lambda_i(h) \neq \lambda_j(h)$ for $i \neq j$.

From equations 6 and 7, $\pi(h)u - \lambda_1(h)u = \sum_{i=2}^{n} (\lambda_i(h) - \lambda_1(h))v_{\lambda_i}$, where each term is not 0 by assumption.

By the inductive assumption, $v_{\lambda_i} \in U$ for $i \geq 2$, hence also $v_{\lambda_1} \in U$. □

Now for the proof of the theorem.

Proof. We want to prove that if $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$, then $\mathfrak{a} = 0$. Note that since $\mathfrak{a}$ is an ideal, it is invariant with respect to **ad** h on $\mathfrak{g}$. Hence by the lemma, either $\mathfrak{g}_\alpha$ is in $\mathfrak{a}$ for some α or $\mathfrak{h} \cap \mathfrak{a}$ is non-zero. This uses the fact that $dim\mathfrak{g}_\alpha = 1$. In the first case, $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{F}h_\alpha \subset \mathfrak{a}$, but $[h_\alpha, \mathfrak{g}_\alpha] \neq 0$ since $\alpha(h_\alpha) \neq 0$. So $\mathfrak{a}$ contains the non-abelian subalgebra $\mathbb{F}h_\alpha \oplus \mathfrak{g}_\alpha$, which is impossible since $\mathfrak{a}$ is abelian.

In the second case, where $\mathfrak{h} \cap \mathfrak{a}$ is non-zero, let $h \in \mathfrak{a}$, $h \in \mathfrak{h}$, $h \neq 0$. By condition 3, $\mathfrak{h}^*$ is spanned by Δ, so $\alpha(h) \neq 0$ for some $\alpha \in \Delta$. Hence $[h, \mathfrak{g}_\alpha] = \mathfrak{g}_\alpha \in \mathfrak{a}$, which again contradicts that $\mathfrak{a}$ is abelian. So a=0.

This proves that $\mathfrak{g}$ is semisimple.

Now the proof of simplicity, given that Δ is indecomposable.

Since $\mathfrak{g}$ is semisimple, we have that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where the $\mathfrak{g}_i$ are non-zero ideals. In the contrary case, by our discussion before Definition 14.2, this implies that Δ is decomposable. This is a contradiction, so $\mathfrak{g}$ is simple. □

The following is an easy method to determine if a set Δ is indecomposable.

Exercise 14.2. Prove that a finite subset: $\Delta \subset V \setminus \{0\}$ is an indecomposable set if and only if for any $\alpha, \beta \in \Delta$, there exists a sequence $\gamma_1, \gamma_2 \ldots \gamma_s$ such that $\alpha = \gamma_1$, $\beta = \gamma_s$ and $\gamma_i + \gamma_{i+1} \in \Delta$ for $i = 1 \ldots s-1$.

Also for any $\Delta \subset V \setminus \{0\}$, construct its canonical decomposition into a disjoint union of indecomposable sets.

Proof. Consider the contrapositive. Suppose Δ is decomposable. Therefore, $\Delta = \Delta_1 \cup \Delta_2$. Take some $\alpha \in \Delta_1, \beta \in \Delta_2$. No sequence can exist with $\gamma_1 = \alpha$ and $\gamma_s = \beta$. For at some step in any such sequence $\gamma_i \in \Delta_1$ and $\gamma_{i+1} \in \Delta_2$. But $\gamma_1 + \gamma_2 \in \Delta$ contradicts the definition of a decomposition.

Suppose for some $\alpha, \beta \in \Delta$ no sequence exists. Let Δ_1 be the set of roots for which a sequence with α exists. Let Δ_2 be the set of roots for which a sequence with α does not exist. This is clearly a disjoint partition of Δ. Further Δ_1 and Δ_2 are non-empty as $\alpha \in \Delta_1$ $\beta \in \Delta_2$. For $\alpha' \in \Delta_1$, $\beta' \in \Delta_2$ $\alpha' + \beta' \notin \Delta$ as otherwise one can concatenate the sequence from $\alpha = \gamma_1$ to $\alpha' = \gamma_s$ and from $\alpha' = \gamma_s$ to $\beta' = \gamma_{s+1}$. Thus, $\Delta_1 \sqcup \Delta_2$, where Δ_1 is indecomposable. Now apply the same argument to Δ_2, etc. Since Δ is a finite set, we obtain the decomposition of Δ in a finite number of steps. □

14.1 Examples of Semisimple Lie Algebras

Example 14.1. $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field with characteristic 0. Let the Cartan subalgebra $\mathfrak{h}$ be the set of all traceless diagonal matrices. $\mathfrak{h}$ lies in D_n, the space of all diagonal matrices. Denote by ϵ_i the following linear function on D: $\epsilon_i(A) = a_i$, the ith coordinate of the diagonal of matrix A. The set $\{\epsilon_i \mid i = 1 \ldots n\}$ is clearly a basis of D^*. If $char\mathbb{F} \nmid n$, another basis is $\{\epsilon_i - \epsilon_{i+1}, \epsilon_1 + \ldots + \epsilon_n \mid i = 1 \ldots n-1\}$, since it generates the first basis and has the same size. $\mathfrak{h} = \{a \in D \mid (\epsilon_1 + \ldots + \epsilon_n)a = 0\}$. Therefore, $\mathfrak{h}^* = D^*/(\epsilon_1 + \ldots + \epsilon_n)$, and $\{\epsilon_i - \epsilon_{i+1} \mid i \neq j\}$ is a basis for h^*.

The root space decomposition of $\mathfrak{g} = \mathfrak{sl}_n(F)$ is $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$, where $\Delta = \{\epsilon_i - \epsilon_j | i \neq j\}$ and $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{F} e_{ij}$.

To prove $\mathfrak{sl}_n(\mathbb{F})$ is semisimple, we check the conditions of theorem 14.2. 1. is clearly true. 2. $[e_{ij}, e_{ji}] = e_{ii} - e_{jj}, (\epsilon_i - \epsilon_j)(e_{ii} - e_{jj}) = 2 \neq 0$. This is why $\mathbb{F}$ cannot be characteristic 2. 3. When $char \nmid n$, the $\epsilon_i - \epsilon_j$ span $\mathfrak{h}$, so Δ spans $\mathfrak{h}$.

This implies $\mathfrak{sl}_n(\mathbb{F})$ is semisimple for any $n \neq 2$ and $char\mathbb{F} \nmid n$. To show it is simple, we prove that Δ is indecomposable.

Let $\alpha = \epsilon_i - \epsilon_j$, $\beta = \epsilon_s - \epsilon_t$. Using exercise 14.2, let $\gamma_1 = \alpha$, $\gamma_2 = \epsilon_j - \epsilon_s$, $\gamma_3 = \beta$. This is a string from α to β as $\gamma_1 - \gamma_2 = \epsilon_i - \epsilon_s \in \Delta$ and $\gamma_2 - \gamma_3 = \epsilon_j - \epsilon_t \in \Delta$. This proves Δ is indecomposable and that $\mathfrak{sl}_n(\mathbb{F})$ is simple.

Exercise 14.3. The above argument fails if $char\mathbb{F} \mid n$. As $\mathfrak{sl}_n(\mathbb{F})$ contains a non-trivial abelian ideal, $Z(\mathfrak{sl}_n(\mathbb{F}))$ since $I_n \in Z(\mathfrak{sl}_n(\mathbb{F}))$. How does the argument fail?

Proof. The argument fails because Δ does not span $\mathfrak{h}^*$. When $char\mathbb{F} \mid n$, $I_n \in \mathfrak{h}$ and Δ is still $\{\epsilon_i - \epsilon_j \mid i \neq j\}$. Since $(\epsilon_i - \epsilon_j)(I_n) = 0$, Δ can't span $\mathfrak{h}^*$. □