2.5 Tangent, Normal and Binormal Vectors

Three vectors play an important role when studying the motion of an object along a space curve. These vectors are the unit tangent vector, the principal normal vector and the binormal vector. We have already defined the unit tangent vector. In this section, we define the other two vectors.

Let us start by reviewing the definition of the unit tangent vector.

Definition 159 (Unit Tangent Vector) *Let C be a smooth curve with position vector $\overrightarrow{r}(t)$. The **unit tangent vector**, denoted $\overrightarrow{T}(t)$ is defined to be*

$$\overrightarrow{T}(t) = \frac{\overrightarrow{r}'(t)}{\|\overrightarrow{r}'(t)\|}$$

2.5.1 Normal and Binormal Vectors

Definition 160 (Normal Vector) *Let C be a smooth curve with position vector $\overrightarrow{r}(t)$. The **principal unit normal** vector or simply the **normal** vector, denoted $\overrightarrow{N}(t)$ is defined to be:*

$$\overrightarrow{N}(t) = \frac{\overrightarrow{T}'(t)}{\left\|\overrightarrow{T}'(t)\right\|} \tag{2.15}$$

The name of this vector suggests that it is normal to something, the question is to what? By definition, $\overrightarrow{T}$ is a unit vector, that is $\left\|\overrightarrow{T}(t)\right\| = 1$. From proposition 125, it follows that $\overrightarrow{T}'(t) \perp \overrightarrow{T}(t)$. Thus, $\overrightarrow{N}(t) \perp \overrightarrow{T}(t)$. In fact, $\overrightarrow{N}(t)$ is a unit vector, perpendicular to $\overrightarrow{T}$ pointing in the direction where the curve is bending.

Proposition 161 *Let C be a smooth curve with position vector $\overrightarrow{r}(t)$. Then,*

$$\begin{aligned}\overrightarrow{N}(t) &= \frac{1}{\kappa}\frac{\overrightarrow{T}'(t)}{\|\overrightarrow{r}'(t)\|} \\ &= \frac{1}{\kappa}\frac{d\overrightarrow{T}}{ds}\end{aligned}$$

Proof. *We know that $\overrightarrow{N}(t) = \dfrac{\overrightarrow{T}'(t)}{\left\|\overrightarrow{T}'(t)\right\|}$ and $\kappa = \frac{\left\|\overrightarrow{T}'(t)\right\|}{\|\overrightarrow{r}'(t)\|}$. So, we have $\left\|\overrightarrow{T}'(t)\right\| = \kappa\|\overrightarrow{r}'(t)\|$ hence*

$$\begin{aligned}\frac{\overrightarrow{T}'(t)}{\left\|\overrightarrow{T}'(t)\right\|} &= \frac{\overrightarrow{T}'(t)}{\kappa\|\overrightarrow{r}'(t)\|} \\ &= \frac{1}{\kappa}\frac{\overrightarrow{T}'(t)}{\|\overrightarrow{r}'(t)\|}\end{aligned}$$

Earlier, we sat that $\frac{d\overrightarrow{T}}{ds} = \frac{\overrightarrow{T}'(t)}{\|\overrightarrow{r}'(t)\|}$ *hence the second equality.* ∎

Definition 162 (Binormal Vector) *Let* C *be a smooth curve with position vector* $\overrightarrow{r}(t)$*. The* ***binormal vector****, denoted* $\overrightarrow{B}(t)$*, is defined to be*

$$\overrightarrow{B}(t) = \overrightarrow{T}(t) \times \overrightarrow{N}(t)$$

Since both $\overrightarrow{T}(t)$ *and* $\overrightarrow{N}(t)$ *are unit vectors and perpendicular, it follows that* $\overrightarrow{B}(t)$ *is also a unit vector. It is perpendicular to both* $\overrightarrow{T}(t)$ *and* $\overrightarrow{N}(t)$*.*

Example 163 *Consider the circular helix* $\overrightarrow{r}(t) = \langle \cos t, \sin t, t \rangle$*. Find the unit tangent, normal and binormal vectors.*

- *Unit Tangent: Since* $\overrightarrow{T}(t) = \dfrac{\overrightarrow{r}'(t)}{\|\overrightarrow{r}'(t)\|}$*, we need to compute* $\overrightarrow{r}'(t)$ *and* $\|\overrightarrow{r}'(t)\|$*.*

$$\overrightarrow{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$$

and

$$\begin{aligned} \overrightarrow{r}'(t) &= \sqrt{\sin^2 t + \cos^2 t + 1} \\ &= \sqrt{2} \end{aligned}$$

Thus

$$\overrightarrow{T}(t) = \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

- *Normal: Since* $\overrightarrow{N}(t) = \dfrac{\overrightarrow{T}'(t)}{\left\|\overrightarrow{T}'(t)\right\|}$*, we need to compute* $\overrightarrow{T}'(t)$ *and* $\left\|\overrightarrow{T}'(t)\right\|$*.*

$$\overrightarrow{T}'(t) = \left\langle \frac{-\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, 0 \right\rangle$$

and

$$\begin{aligned} \left\|\overrightarrow{T}'(t)\right\| &= \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2}} \\ &= \frac{1}{\sqrt{2}} \end{aligned}$$

Thus

$$\overrightarrow{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$$

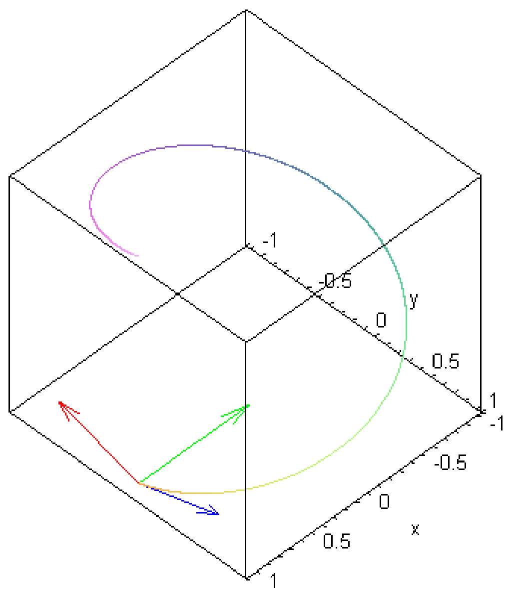


Figure 2.7: Helix and the vectors $\overrightarrow{T}(0)$, $\overrightarrow{N}(0)$ and $\overrightarrow{B}(0)$

- *Binormal:*

$$\begin{aligned}\overrightarrow{B}(t) &= \overrightarrow{T}(t) \times \overrightarrow{N}(t) \\ &= \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \times \langle -\cos t, -\sin t, 0 \rangle \\ \overrightarrow{B}(t) &= \left\langle \frac{\sin t}{\sqrt{2}}, \frac{-\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle\end{aligned}$$

The pictures below (figures 2.7, 2.8 and 2.9) show the helix for $t \in [0, 2\pi]$ as well as the three vectors $\overrightarrow{T}(t)$, $\overrightarrow{N}(t)$ and $\overrightarrow{B}(t)$ plotted for various values of t. If the three vectors do not appear to be exactly orthogonal, it is because the scale is not the same in the x, y and z directions.

2.5.2 Osculating and Normal Planes

Definition 164 (Osculating and Normal Planes) *Let C be a smooth curve with position vector $\overrightarrow{r}(t)$. Let P be a point on the curve corresponding to $\overrightarrow{r}(t_0)$ for some value of t.*

1. *The plane through P determined by $\overrightarrow{N}(t_0)$ and $\overrightarrow{B}(t_0)$ is called the **normal plane** of C at P. Note that its normal will be $\overrightarrow{T}(t)$.*

2. *The plane through P determined by $\overrightarrow{T}(t_0)$ and $\overrightarrow{N}(t_0)$ is called the **osculating plane** of C at P. Note that its normal will be $\overrightarrow{B}(t)$.*

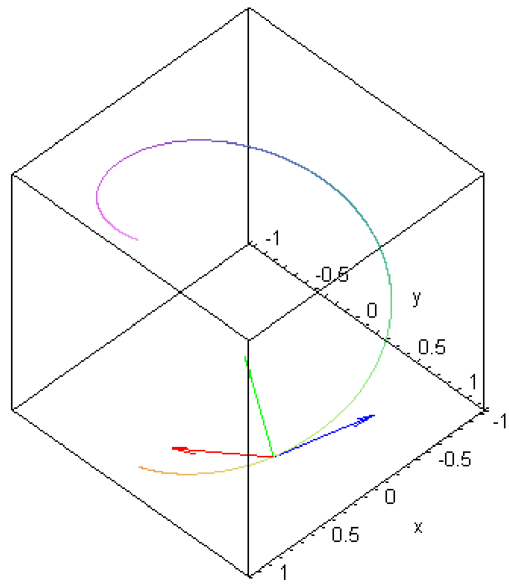


Figure 2.8: Helix and the vectors $\overrightarrow{T}(1)$, $\overrightarrow{N}(1)$ and $\overrightarrow{B}(1)$

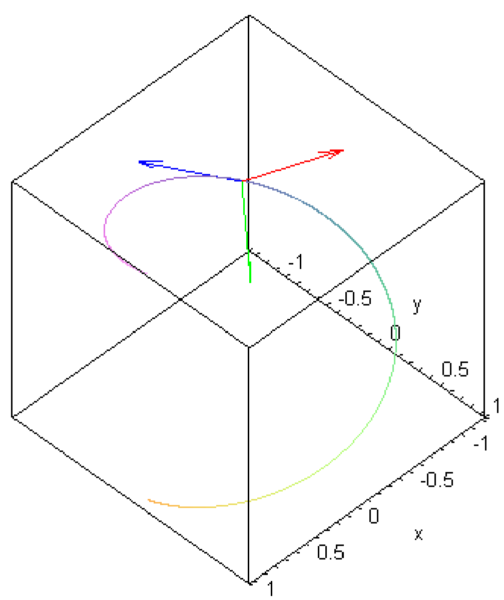


Figure 2.9: Helix and the vectors $\overrightarrow{T}(4)$, $\overrightarrow{N}(4)$ and $\overrightarrow{B}(4)$

3. *The **osculating circle** or the **circle of curvature** at P is the circle which has the following properties:*
   (a) *It lies on the osculating plane.*
   (b) *Has the same tangent at P as C.*
   (c) *Its radius is $\frac{1}{\kappa}$ where κ is computed at P.*
   (d) *Lies on the side of C where $\overrightarrow{N}$ is pointing.*

Example 165 *Find the normal and osculating planes to the helix given by $\overrightarrow{r}(t) = \langle \cos t, \sin t, t \rangle$ at the point $\left(0, 1, \frac{\pi}{2}\right)$.*
Earlier, we found that

$$\overrightarrow{T}(t) = \left\langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\overrightarrow{N}(t) = \langle -\cos t, -\sin t, 0 \rangle$$

and

$$\overrightarrow{B}(t) = \left\langle \frac{\sin t}{\sqrt{2}}, \frac{-\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

At the point $\left(0, 1, \frac{\pi}{2}\right)$, that is when $t = \frac{\pi}{2}$, we have

$$\overrightarrow{T}\left(\frac{\pi}{2}\right) = \left\langle \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

$$\overrightarrow{N}\left(\frac{\pi}{2}\right) = \langle 0, -1, 0 \rangle$$

and

$$\overrightarrow{B}\left(\frac{\pi}{2}\right) = \left\langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$$

- *Normal Plane: It is the plane through $\left(0, 1, \frac{\pi}{2}\right)$ with normal $\overrightarrow{T}\left(\frac{\pi}{2}\right) = \left\langle \frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right\rangle$. Thus its equation is*

$$\frac{-1}{\sqrt{2}}(x - 0) + \frac{1}{\sqrt{2}}\left(z - \frac{\pi}{2}\right) = 0$$

Multiplying each side by $\sqrt{2}$ gives

$$-(x - 0) + \left(z - \frac{\pi}{2}\right) = 0$$

or

$$z - x = \frac{\pi}{2}$$

- *Osculating Plane: It is the plane through* $\left(0,1,\frac{\pi}{2}\right)$ *with normal* $\overrightarrow{B}\left(\frac{\pi}{2}\right)=\left\langle\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right\rangle$ *Thus its equation is*

$$\frac{1}{\sqrt{2}}(x-0)+\frac{1}{\sqrt{2}}\left(z-\frac{\pi}{2}\right)=0$$

Multiplying each side by $\sqrt{2}$ *gives*

$$(x-0)+\left(z-\frac{\pi}{2}\right)=0$$

or

$$x+z=\frac{\pi}{2}$$

Example 166 *Find and graph the osculating circle for the parabola* $y=x^2$ *at the origin.*
We need to find $\overrightarrow{r}'(t)$*,* $\overrightarrow{N}(t)$ *and* κ*. Recall that* $\overrightarrow{r}(t)=\left\langle t,t^2\right\rangle$

$$\overrightarrow{r}'(t)=\langle 1,2t\rangle$$

At the origin, $t=0$ *hence* $\overrightarrow{r}'(0)=\langle 1,0\rangle=\overrightarrow{i}$*. Using formula 2.14, we see that*

$$\kappa(x)=\frac{2}{(1+4x^2)^{\frac{3}{2}}}$$

Hence $\kappa(0)=2$*. Also,* $\overrightarrow{N}(t)=\frac{\overrightarrow{T}'(t)}{\left\|\overrightarrow{T}'(t)\right\|}$.

$$\begin{aligned}\overrightarrow{T}(t) &= \frac{\overrightarrow{r}'(t)}{\|\overrightarrow{r}'(t)\|}\\ &= \left\langle\frac{1}{1+4t^2},\frac{2t}{1+4t^2}\right\rangle\end{aligned}$$

Thus,

$$\begin{aligned}\overrightarrow{T}'(t) &= \left\langle\frac{-8t}{(1+4t^2)^2},\frac{2\left(1+4t^2-2t(8t)\right)}{(1+4t^2)^2}\right\rangle\\ &= \left\langle\frac{-8t}{(1+4t^2)^2},\frac{2-8t^2}{(1+4t^2)^2}\right\rangle\end{aligned}$$

and Thus

$$\begin{aligned}\left\|\overrightarrow{T}'(t)\right\| &= \frac{1}{\left(1+4t^2\right)^2}\sqrt{64t^2+4+64t^4-32t^2} \\ &= \frac{1}{\left(1+4t^2\right)^2}\sqrt{4+64t^4+32t^2} \\ &= \frac{\sqrt{4\left(4t^2+1\right)^2}}{\left(1+4t^2\right)^2} \\ &= \frac{2}{\left(1+4t^2\right)}\end{aligned}$$

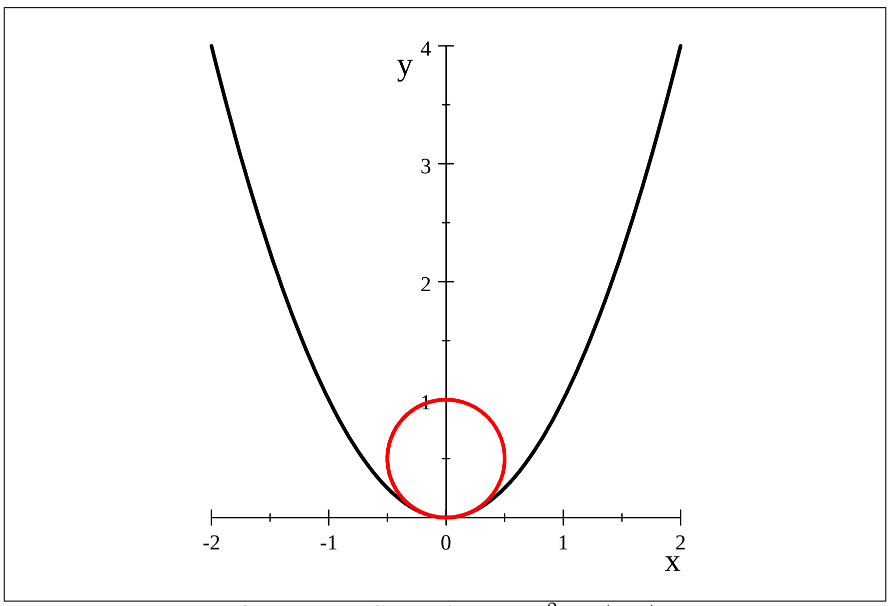
So,

$$\begin{aligned}\overrightarrow{N}(t) &= \left\langle \frac{\frac{-8t}{(1+4t^2)^2}}{\frac{2}{(1+4t^2)}}, \frac{\frac{2-8t^2}{(1+4t^2)^2}}{\frac{2}{(1+4t^2)}} \right\rangle \\ &= \left\langle \frac{-4t}{1+4t^2}, \frac{1-4t^2}{1+4t^2} \right\rangle\end{aligned}$$

Thus

$$\overrightarrow{N}(0) = \langle 0,1\rangle = \overrightarrow{j}$$

Hence, the osculating circle is a circle of radius $\frac{1}{2}$. *Its center is* $\frac{1}{2}$ *units from the origin, in the direction of* $\overrightarrow{j}$ *hence the center is* $\left(0,\frac{1}{2}\right)$. *Thus, the circle is* $x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4}$. *The graph of* $y=x^2$ *and its osculating circle at the origin are shown in figure 166.*



Osculating Circle for $y=x^2$ *at* $(0,0)$

Make sure you have read, studied and understood what was done above before attempting the problems.

2.5.3 Problems

1. Find $\overrightarrow{N}(t)$ in the odd # 1-3 and 9-13 at the end of section 10.4 in your book.

2. # 19, 21, 23, 25 at the end of section 10.4 in your book.