7-th Canadian Mathematical Olympiad 1975

1. Simplify
$$\left(\frac{1\cdot 2\cdot 4+2\cdot 4\cdot 8+\cdots+n\cdot 2n\cdot 4n}{1\cdot 3\cdot 9+2\cdot 6\cdot 18+\cdots+n\cdot 3n\cdot 9n}\right)^{1/3}.$$

2. A sequence $a_1, a_2, a_3, \ldots$ satisfies

   (a) $a_1 = 1/2$;

   (b) $a_1 + a_2 + \cdots + a_n = n^2 a_n$ for each $n \geq 1$.

   Determine the value of a_n.

3. Indicate on the $xy-$plane the set of points (x,y) for which $[x]^2 + [y]^2 = 4$.

4. Find a positive number such that its decimal part, its integral part and the number itself ($x-[x]$, $[x]$ and x) form a geometric progression.

5. Let $ABCD$ be a convex quadrilateral inscribed in a circle and P,Q,R,S be the midpoints of the arcs AB,BC,CD,DA, respectively. Prove that PR is perpendicular to QS.

6. (a) 15 chairs are equally placed around a circular table with name cards for 15 guests. The guests fail to notice these cards, and it turns out that no one is sitting in the correct seat. Prove that the table can be rotated in such a way that at least two guests are correctly seated.

   (b) Give an example of an arrangement in which just one of the 15 guests is correctly seated and for which no rotation correctly places two guests.

7. Is the function $\sin x^2$ periodic? Prove your assertion.

8. Given a positive integer k, find all polynomials $P(x)$ with real coefficients such that $P(P(x)) = P(x)^k$.

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