

# Kepler's Laws of Planetary Motion

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## Abstract

Kepler worked from 1601 to 1612 in Prague as the Imperial Mathematician. Taking over from Tycho Brahe, and using the tremendous amount of data gathered by Brahe from naked-eye astronomical observation, he formulated three laws governing planetary motion. Fortunately, among his tasks was the study of the planet Mars, whose orbit is quite unlike a circle, at least relatively speaking. This forced Kepler to consider other possibilities and ultimately led to his discovery of elliptical orbits. These laws, which were the first “natural laws” in the modern sense, served to divorce astronomy from theology and philosophy and marry it to physics. At last, the planets were viewed as material bodies, not unlike earth, floating freely in space and moved by physical forces acting on them. Although the theology and philosophy of the time dictated uniform planetary motion and circular orbits, nature was now free to ignore these demands; motion of the planets could be non-uniform and the orbits other than circular. These laws, particularly the third one, provided strong evidence for Newton's law of universal gravitation.

## 1 Preliminaries

Kepler worked from 1601 to 1612 in Prague as the Imperial Mathematician. Taking over from Tycho Brahe, and using the tremendous amount of data gathered by Brahe from naked-eye astronomical observation, he formulated three laws governing planetary motion. Fortunately, among his tasks was the study of the planet Mars, whose orbit is quite unlike a circle, at least relatively speaking. This forced Kepler to consider other possibilities and ultimately led to his discovery of elliptic orbits.

These laws, which were the first “natural laws” in the modern sense, served to divorce astronomy from theology and philosophy and marry it to physics. At last, the planets were viewed as material bodies, not unlike earth, floating freely in space and moved by physical forces acting on them. Although the theology and philosophy of the time dictated uniform planetary motion and circular orbits, nature was now free to ignore these demands; motion of the planets could be non-uniform and the orbits other than circular. These laws, particularly the third one, provided strong evidence for Newton’s law of universal gravitation. Although the second law preceded the first, Kepler’s Laws are usually enumerated as follows:

- 1. the planets travel around the sun not in circles but in elliptical orbits, with the sun at one focal point;
- 2. a planet’s speed is not uniform, but is such that the line segment from the sun to the planet sweeps out equal areas in equal time intervals; and, finally,
- 3. for all the planets, the time required for the planet to complete one orbit around the sun, divided by the  $3/2$  power of its average distance from the sun, is the same constant.

These laws, particularly the third one, provided strong evidence for Newton’s law of universal gravitation. How Kepler discovered these laws without the aid of analytic geometry and differential calculus, with no notion of momentum, and only a vague conception of gravity, is a fascinating story, perhaps best told by Koestler in [4].

Around 1684, Newton was asked by Edmund Halley, of Halley’s comet fame, what the path would be for a planet moving around the sun, if the force of gravity fell off as the square of the distance from the sun. Newton responded that it would be an ellipse. Kepler had already declared that planets moved along elliptical orbits with the sun at one focal point, but his findings were based on observation and imagination, not deduction from physical principles. Halley asked Newton to provide a proof. To supply such a proof, Newton needed to write a whole book, the *Principia*, published in 1687, in which he had to deal with such mathematically difficult questions as what the gravitational force is on a point when the attracting body is not just another point, but a sphere, like the sun.

With the help of vector calculus, a later invention, Kepler’s laws can be derived as consequences of Newton’s inverse square law for gravitational attraction.

## 2 A Little Vector Calculus

We consider a body with constant mass  $m$  moving through three-dimensional space along a curve

$$\mathbf{r}(t) = (x(t), y(t), z(t)),$$

where  $t$  is time and the sun is the origin. The velocity vector at time  $t$  is then

$$\mathbf{v}(t) = \mathbf{r}'(t) = (x'(t), y'(t), z'(t)),$$

and the acceleration vector at time  $t$  is

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = (x''(t), y''(t), z''(t)).$$

The *linear momentum* vector is

$$\mathbf{p}(t) = m\mathbf{v}(t).$$

One of the most basic laws of motion is that the vector  $\mathbf{p}'(t) = m\mathbf{v}'(t) = m\mathbf{a}(t)$  is equal to the external force exerted on the body. When a body, or more precisely, the center of mass of the body, does not change location, all it can do is rotate. In order for a body to rotate about an axis a *torque* is required. Just as work equals force times distance moved, work done in rotating a body equals torque times angle through which it is rotated. Just as force is the time derivative of  $\mathbf{p}(t)$ , the linear momentum vector, we find that torque is the time derivative of something else, called the *angular momentum vector*.

## 3 Torque and Angular Momentum

Consider a body rotating around the origin in two-dimensional space, whose position at time  $t$  is

$$\mathbf{r}(t) = (r \cos \theta(t), r \sin \theta(t)).$$

Then at time  $t + \Delta t$  it is at

$$\mathbf{r}(t + \Delta t) = (r \cos(\theta(t) + \Delta\theta), r \sin(\theta(t) + \Delta\theta)).$$

Therefore, using trig identities, we find that the change in the  $x$ -coordinate is approximately

$$\Delta x = -r\Delta\theta \sin \theta(t) = -y(t)\Delta\theta,$$

and the change in the  $y$ -coordinate is approximately

$$\Delta y = r\Delta\theta \cos\theta(t) = x(t)\Delta\theta.$$

The infinitesimal work done by a force  $\mathbf{F} = (F_x, F_y)$  in rotating the body through the angle  $\Delta\theta$  is then approximately

$$\Delta W = F_x\Delta x + F_y\Delta y = (F_yx(t) - F_xy(t))\Delta\theta.$$

Since work is torque times angle, we define the torque to be

$$\tau = F_yx(t) - F_xy(t).$$

The entire motion is taking place in two dimensional space. Nevertheless, it is convenient to make use of the concept of cross product of three-dimensional vectors to represent the torque. When we rewrite

$$\mathbf{r}(t) = (x(t), y(t), 0),$$

and

$$\mathbf{F} = (F_x, F_y, 0),$$

we find that

$$\mathbf{r}(t) \times \mathbf{F} = (0, 0, F_yx(t) - F_xy(t)) = (0, 0, \tau) = \tau.$$

Now we use the fact that the force is the time derivative of the vector  $\mathbf{p}(t)$  to write

$$\tau = (0, 0, \tau) = \mathbf{r}(t) \times \mathbf{p}'(t).$$

**Exercise 3.1** *Show that*

$$\mathbf{r}(t) \times \mathbf{p}'(t) = \frac{d}{dt}(\mathbf{r}(t) \times \mathbf{p}(t)). \quad (3.1)$$

By analogy with force as the time derivative of linear momentum, we define torque as the time derivative of angular momentum, which, from the calculations just performed, leads to the definition of the *angular momentum vector* as

$$\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{p}(t).$$

We need to say a word about the word “vector”. In our example of rotation in two dimensions we introduced the third dimension as merely a notational convenience. It

is convenient to be able to represent the torque as  $\mathbf{L}'(t) = (0, 0, \tau)$ , but when we casually call  $L(t)$  the angular momentum vector, physicists would tell us that we haven't yet shown that angular momentum is a “vector” in the physicists' sense. Our example was too simple, they would point out. We had rotation about a single fixed axis that was conveniently chosen to be one of the coordinate axes in three-dimensional space. But what happens when the coordinate system changes?

Clearly, they would say, physical objects rotate and have angular momentum. The earth rotates around an axis, but this axis is not always the same axis; the axis wobbles. A well thrown football rotates around its longest axis, but this axis changes as the ball flies through the air. Can we still say that the angular momentum can be represented as

$$\mathbf{L}(t) = \mathbf{r}(t) \times \mathbf{p}(t)?$$

In other words, we need to know that the torque is still the time derivative of  $\mathbf{L}(t)$ , even as the coordinate system changes. In order for something to be a “vector” in the physicists' sense, it needs to behave properly as we switch coordinate systems, that is, it needs to *transform as a vector* [2]. In fact, all is well. This definition of  $\mathbf{L}(t)$  holds for bodies moving along more general curves in three-dimensional space, and we can go on calling  $\mathbf{L}(t)$  the angular momentum vector. Now we begin to exploit the special nature of the gravitational force.

## 4 Gravity is a Central Force

We are not interested here in arbitrary forces, but in the gravitational force that the sun exerts on the body, which has special properties that we shall exploit. In particular, this gravitational force is a *central force*.

**Definition 4.1** *We say that the force is a central force if*

$$\mathbf{F}(t) = h(t)\mathbf{r}(t),$$

*for each  $t$ , where  $h(t)$  denotes a scalar function of  $t$ ; that is, the force is central if it is proportional to  $\mathbf{r}(t)$  at each  $t$ .*

**Proposition 4.1** *If  $\mathbf{F}(t)$  is a central force, then  $\mathbf{L}'(t) = \mathbf{0}$ , for all  $t$ , so that  $\mathbf{L} = \mathbf{L}(t)$  is a constant vector and  $L = \|\mathbf{L}(t)\| = \|\mathbf{L}\|$  is a constant scalar, for all  $t$ .*

**Proof:** From Equation (3.1) we have

$$\mathbf{L}'(t) = \mathbf{r}(t) \times \mathbf{p}'(t) = \mathbf{r}(t) \times \mathbf{F}(t) = h(t)\mathbf{r}(t) \times \mathbf{r}(t) = \mathbf{0}.$$

We see then that the angular momentum vector  $\mathbf{L}(t)$  is *conserved* when the force is central. ■

**Proposition 4.2** *If  $\mathbf{L}'(t) = \mathbf{0}$ , then the curve  $\mathbf{r}(t)$  lies in a plane.*

**Proof:** We have

$$\mathbf{r}(t) \cdot \mathbf{L} = \mathbf{r}(t) \cdot \mathbf{L}(t) = \mathbf{r}(t) \cdot (\mathbf{r}(t) \times \mathbf{p}(t)),$$

which is the volume of the parallelepiped formed by the three vectors  $\mathbf{r}(t)$ ,  $\mathbf{r}(t)$  and  $\mathbf{p}(t)$ , which is obviously zero. Therefore, for every  $t$ , the vector  $\mathbf{r}(t)$  is orthogonal to the constant vector  $\mathbf{L}$ . So, the curve lies in a plane with normal vector  $\mathbf{L}$ . ■

## 5 The Second Law

We know now that, since the force is central, the curve described by  $\mathbf{r}(t)$  lies in a plane. This allows us to use polar coordinate notation [5]. We write

$$\mathbf{r}(t) = \rho(t)(\cos \theta(t), \sin \theta(t)) = \rho(t)\mathbf{u}_r(t),$$

where  $\rho(t)$  is the length of the vector  $\mathbf{r}(t)$  and

$$\mathbf{u}_r(t) = \frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|} = (\cos \theta(t), \sin \theta(t))$$

is the unit vector in the direction of  $\mathbf{r}(t)$ . We also define

$$\mathbf{u}_\theta(t) = (-\sin \theta(t), \cos \theta(t)),$$

so that

$$\mathbf{u}_\theta(t) = \frac{d}{d\theta} \mathbf{u}_r(t),$$

and

$$\mathbf{u}_r(t) = -\frac{d}{d\theta} \mathbf{u}_\theta(t).$$

**Exercise 5.1** *Show that*

$$\mathbf{p}(t) = m\rho'(t)\mathbf{u}_r(t) + m\rho(t)\frac{d\theta}{dt}\mathbf{u}_\theta(t). \tag{5.1}$$

**Exercise 5.2** *View the vectors  $\mathbf{r}(t)$ ,  $\mathbf{p}(t)$ ,  $\mathbf{u}_r(t)$  and  $\mathbf{u}_\theta(t)$  as vectors in three-dimensional space, all with third component equal to zero. Show that*

$$\mathbf{u}_r(t) \times \mathbf{u}_\theta(t) = \mathbf{k} = (0, 0, 1),$$

for all  $t$ . Use this and Equation (5.1) to show that

$$\mathbf{L} = \mathbf{L}(t) = \left( m\rho(t)^2 \frac{d\theta}{dt} \right) \mathbf{k},$$

so that  $L = m\rho(t)^2 \frac{d\theta}{dt}$ , the moment of inertia times the angular velocity, is constant.

Let  $t_0$  be some arbitrary time, and for any time  $t \geq t_0$  let  $A(t)$  be the area swept out by the planet in the time interval  $[t_0, t]$ . Then  $A(t_2) - A(t_1)$  is the area swept out in the time interval  $[t_1, t_2]$ .

In the very short time interval  $[t, t + \Delta t]$  the vector  $\mathbf{r}(t)$  sweeps out a very small angle  $\Delta\theta$ , and the very small amount of area formed is then approximately

$$\Delta A = \frac{1}{2} \rho(t)^2 \Delta\theta.$$

Dividing by  $\Delta t$  and taking limits, as  $\Delta t \rightarrow 0$ , we get

$$\frac{dA}{dt} = \frac{1}{2} \rho(t)^2 \frac{d\theta}{dt} = \frac{L}{2m}.$$

Therefore, the area swept out between times  $t_1$  and  $t_2$  is

$$A(t_2) - A(t_1) = \int_{t_1}^{t_2} \frac{dA}{dt} dt = \int_{t_1}^{t_2} \frac{L}{2m} dt = \frac{L(t_2 - t_1)}{2m}.$$

This is Kepler's Second Law.

## 6 The First Law

We saw previously that the angular momentum vector is conserved when the force is central. When Newton's inverse-square law holds, there is another conservation law; the *Runge-Lenz vector* is also conserved. We shall use this fact to derive the First Law.

Let  $M$  denote the mass of the sun, and  $G$  Newton's gravitational constant.

**Definition 6.1** *The force obeys Newton's inverse square law if*

$$\mathbf{F}(t) = h(t)\mathbf{r}(t) = -\frac{mMG}{\rho(t)^3}\mathbf{r}(t).$$

Then we can write

$$\mathbf{F}(t) = -\frac{mMG}{\rho(t)^2} \frac{\mathbf{r}(t)}{\|\mathbf{r}(t)\|} = -\frac{mMG}{\rho(t)^2} \mathbf{u}_r(t).$$

**Definition 6.2** *The Runge-Lenz vector is*

$$\mathbf{K}(t) = \mathbf{p}(t) \times \mathbf{L}(t) - k\mathbf{u}_r(t),$$

where  $k = m^2MG$ .

**Exercise 6.1** *Show that the velocity vectors  $\mathbf{r}'(t)$  lie in the same plane as the curve  $\mathbf{r}(t)$ .*

**Exercise 6.2** *Use the rule*

$$\mathbf{A} \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{A} \cdot \mathbf{B})\mathbf{A} - (\mathbf{A} \cdot \mathbf{A})\mathbf{B}$$

to show that  $\mathbf{K}'(t) = \mathbf{0}$ , so that  $\mathbf{K} = \mathbf{K}(t)$  is a constant vector and  $K = \|\mathbf{K}\|$  is a constant scalar.

So the Runge-Lenz vector is conserved when the force obeys Newton's inverse square law.

**Exercise 6.3** *Use the rule in the previous exercise to show that the constant vector  $\mathbf{K}$  also lies in the plane of the curve  $\mathbf{r}(t)$ .*

**Exercise 6.4** *Show that*

$$\mathbf{K} \cdot \mathbf{r}(t) = L^2 - k\rho(t).$$

It follows from this exercise that

$$L^2 - k\rho(t) = \mathbf{K} \cdot \mathbf{r}(t) = K\rho(t) \cos \alpha(t),$$

where  $\alpha(t)$  is the angle between the vectors  $\mathbf{K}$  and  $\mathbf{r}(t)$ . From this we get

$$\rho(t) = L^2 / (k + K \cos \alpha(t)).$$

For  $k > K$ , this is the equation of an ellipse having eccentricity  $e = K/k$ . This is Kepler's First Law.

Kepler initially thought that the orbits were "egg-shaped", but later came to realize that they were ellipses. Although Kepler did not have the analytical geometry tools to help him, he was familiar with the mathematical development of ellipses in the *Conics*, the ancient book by Apollonius, written in Greek in Alexandria about 200 BC. Conics, or conic sections, are the terms used to describe the two-dimensional

curves, such as ellipses, parabolas and hyperbolas, formed when a plane intersects an infinite double cone (think “hour-glass”).

Apollonius was interested in astronomy and Ptolemy was certainly aware of the work of Apollonius, but it took Kepler to overcome the bias toward circular motion and introduce conic sections into astronomy. As related by Bochner [1], there is a bit of mystery concerning Kepler’s use of the *Conics*. He shows that he is familiar with a part of the *Conics* that existed only in Arabic until translated into Latin in 1661, well after his time. How he gained that familiarity is the mystery.

## 7 The Third Law

As the planet moves around its orbit, the closest distance to the sun is

$$\rho_{\min} = L^2/(k + K),$$

and the farthest distance is

$$\rho_{\max} = L^2/(k - K).$$

The average of these two is

$$a = \frac{1}{2}(\rho_{\min} + \rho_{\max}) = 2kL^2/(k^2 - K^2);$$

this is the semi-major axis of the ellipse. The semi-minor axis has length  $b$ , where

$$b^2 = a^2(1 - e^2).$$

Therefore,

$$b = \frac{L\sqrt{a}}{\sqrt{k}}.$$

The area of this ellipse is  $\pi ab$ . But we know from the first law that the area of the ellipse is  $\frac{L}{2m}$  times the time  $T$  required to complete a full orbit. Equating the two expressions for the area, we get

$$T^2 = \frac{4\pi^2}{MG}a^3.$$

This is the third law.

The first two laws deal with the behavior of one planet; the third law is different. The third law describes behavior that is common to all the planets in the solar system, thereby suggesting a universality to the force of gravity.

## 8 Dark Matter and Dark Energy

Ordinary matter makes up only a small fraction of the “stuff” in the universe. About 25 percent of the stuff is *dark matter* and over two thirds is *dark energy*. Because neither of these interacts with electromagnetic radiation, evidence for their existence is indirect.

Suppose, for the moment, that a planet moves in a circular orbit of radius  $a$ , centered at the sun. The orbital time is  $T$ , so, by Kepler’s third law, the speed with which the planet orbits the sun is  $\sqrt{\frac{MG}{a}}$ , so the farther away the planet the slower it moves. Spiral galaxies are like large planetary systems, with some stars nearer to the center of the galaxy than others. We would expect those stars farther from the center of mass of the galaxy to be moving more slowly, but this is not the case. One explanation for this is that there is more mass present, *dark mass* we cannot detect, spread throughout the galaxy and not concentrated just near the center.

According to Einstein, massive objects can bend light. This *gravitational lensing*, distorting the light from distant stars, has been observed by astronomers, but cannot be simply the result of the observable mass present; there must be more mass out there. Again, this provides indirect evidence for dark mass.

The universe is expanding. Until fairly recently, it was believed that, although it was expanding, the rate of expansion was decreasing; the mass in the universe was exerting gravitational pull that was slowing down the rate of expansion. The question was whether or not the expansion would eventually stop and contraction begin. When the rate of expansion was measured, it was discovered that the rate was increasing, not decreasing. The only possible explanation for this seemed to be that *dark energy* was operating and with sufficient strength to overcome not just the pull of ordinary matter, but of the dark matter as well. Understanding dark matter and dark energy is one of the big challenges for physicists of the twenty-first century.

## 9 From Kepler to Newton

Our goal, up to now, has been to show how Kepler’s three laws can be derived from Newton’s inverse-square law, which, of course, is not how Kepler obtained the laws. Kepler arrived at his laws empirically, by studying the astronomical data. Newton was aware of Kepler’s laws and they influenced his work on universal gravitation. When asked what would explain Kepler’s elliptical orbits, Newton replied that he had calculated that an inverse-square law would do it. Newton found that the force

required to cause the moon to deviate from a tangent line was approximately that given by an inverse-square fall-off in gravity.

It is interesting to ask if the inverse-square law can be derived from Kepler's three laws; the answer is yes, as we shall see in this section. What follows is taken from [3].

We found previously that

$$\frac{dA}{dt} = \frac{1}{2}\rho(t)^2 \frac{d\theta}{dt} = \frac{L}{2m} = c. \quad (9.1)$$

Differentiating with respect to  $t$ , we get

$$\rho(t)\rho'(t) \frac{d\theta}{dt} + \frac{1}{2}\rho(t)^2 \frac{d^2\theta}{dt^2} = 0, \quad (9.2)$$

so that

$$2\rho'(t) \frac{d\theta}{dt} + \rho(t) \frac{d^2\theta}{dt^2} = 0. \quad (9.3)$$

From this, we shall prove that the force is central, directed towards the sun.

As we did earlier, we write the position vector  $\mathbf{r}(t)$  as

$$\mathbf{r}(t) = \rho(t)\mathbf{u}_r(t),$$

so, suppressing the dependence on the time  $t$ , and using the identities

$$\frac{d\mathbf{u}_r}{dt} = \mathbf{u}_\theta \frac{d\theta}{dt},$$

and

$$\frac{d\mathbf{u}_\theta}{dt} = -\mathbf{u}_r \frac{d\theta}{dt},$$

we write the velocity vector as

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\rho}{dt}\mathbf{u}_r + \rho \frac{d\mathbf{u}_r}{dt} = \frac{d\rho}{dt}\mathbf{u}_r + \rho \frac{d\mathbf{u}_r}{d\theta} \frac{d\theta}{dt} = \frac{d\rho}{dt}\mathbf{u}_r + \rho \frac{d\theta}{dt}\mathbf{u}_\theta,$$

and the acceleration vector as

$$\begin{aligned} \mathbf{a} &= \frac{d^2\rho}{dt^2}\mathbf{u}_r + \frac{d\rho}{dt} \frac{d\mathbf{u}_r}{dt} + \frac{d\rho}{dt} \frac{d\theta}{dt}\mathbf{u}_\theta + \rho \frac{d^2\theta}{dt^2}\mathbf{u}_\theta + \rho \frac{d\theta}{dt} \frac{d\mathbf{u}_\theta}{dt} \\ &= \frac{d^2\rho}{dt^2}\mathbf{u}_r + \frac{d\rho}{dt} \frac{d\theta}{dt}\mathbf{u}_\theta + \frac{d\rho}{dt} \frac{d\theta}{dt}\mathbf{u}_\theta + \rho \frac{d^2\theta}{dt^2}\mathbf{u}_\theta - \rho \frac{d\theta}{dt} \frac{d\theta}{dt}\mathbf{u}_r. \end{aligned}$$

Therefore, we have

$$\mathbf{a} = \left( \frac{d^2\rho}{dt^2} - \rho \left( \frac{d\theta}{dt} \right)^2 \right) \mathbf{u}_r + \left( 2 \frac{d\rho}{dt} \frac{d\theta}{dt} + \rho \frac{d^2\theta}{dt^2} \right) \mathbf{u}_\theta.$$

Using Equation (9.2), this reduces to

$$\mathbf{a} = \left( \frac{d^2\rho}{dt^2} - \rho \left( \frac{d\theta}{dt} \right)^2 \right) \mathbf{u}_r, \quad (9.4)$$

which tells us that the acceleration, and therefore the force, is directed along the line joining the planet to the sun; it is a central force.

**Exercise 9.1** Prove the following two identities:

$$\frac{d\rho}{dt} = \frac{d\rho}{d\theta} \frac{d\theta}{dt} = \frac{2c}{\rho^2} \frac{d\rho}{dt} \quad (9.5)$$

and

$$\frac{d^2\rho}{dt^2} = \frac{4c^2}{\rho^4} \frac{d^2\rho}{d\theta^2} - \frac{8c^2}{\rho^5} \left( \frac{d\rho}{d\theta} \right)^2. \quad (9.6)$$

Therefore, we can write the acceleration vector as

$$\mathbf{a} = \left( \frac{4c^2}{\rho^4} \frac{d^2\rho}{d\theta^2} - \frac{8c^2}{\rho^5} \left( \frac{d\rho}{d\theta} \right)^2 - \frac{4c^2}{\rho^3} \right) \mathbf{u}_r.$$

To simplify, we substitute  $u = \rho^{-1}$ .

**Exercise 9.2** Prove that the acceleration vector can be written as

$$\mathbf{a} = \left( 4c^2 u^2 \left( -\frac{1}{u^2} \frac{d^2 u}{d\theta^2} + \frac{2}{u^3} \left( \frac{du}{d\theta} \right)^2 \right) - 8c^2 u^5 \left( -\frac{1}{u^2} \frac{du}{d\theta} \right)^2 - 4c^2 u^3 \right) \mathbf{u}_r,$$

so that

$$\mathbf{a} = -4c^2 u^2 \left( \frac{d^2 u}{d\theta^2} + u \right) \mathbf{u}_r. \quad (9.7)$$

Kepler's First Law tells us that

$$\rho(t) = \frac{L^2}{k + K \cos \alpha(t)} = \frac{a(1 - e^2)}{1 + e \cos \alpha(t)},$$

where  $e = K/k$  and  $a$  is the semi-major axis. Therefore,

$$u = \frac{1 + e \cos \alpha(t)}{a(1 - e^2)}.$$

Using Equation (9.7), we can write the acceleration as

$$\mathbf{a} = -\frac{4c^2}{a(1 - e^2)} u^2 \mathbf{u}_r = -\frac{4c^2}{a(1 - e^2)} r^{-2} \mathbf{u}_r,$$

which tells us that the force obeys an inverse-square law. We still must show that this same law applies to each of the planets, that is, that the constant  $\frac{c^2}{a(1 - e^2)}$  does not depend on the particular planet.

**Exercise 9.3** Show that

$$\frac{c^2}{a(1 - e^2)} = \frac{\pi^2 a^3}{T^2},$$

which is independent of the particular planet, according to Kepler's Third Law.

## 10 Newton's Own Proof of the Second Law

Although Newton invented calculus, he relied on geometry for many of his mathematical arguments. A good example is his proof of Kepler's Second Law.

He begins by imagining the planet at the point 0 in the figure. If there were no force coming from the sun, then, by the principle of inertia, the planet would continue in a straight line, with constant speed. The distance  $\Delta$  from the point 0 to the point 1 is the same as the distance from 1 to 2 and the same as the distance from 2 to 3. The areas of the three triangles formed by the sun and the points 0 and 1, the sun and the points 1 and 2, and the sun and the points 2 and 3 are all equal, since they all equal half of the base  $\Delta$  times the height  $H$ . Therefore, in the absence of a force from the sun, the planet sweeps out equal areas in equal times. Now what happens when there is a force from the sun?

Newton now assumes that  $\Delta$  is very small, and that during the short time it would have taken for the planet to move from 1 to 3 there is a force on the planet, directed toward the sun. Because of the small size of  $\Delta$ , he safely assumes that the direction of this force is unchanged and is directed along the line from 2, the midpoint of 1 and 3, to the sun. The effect of such a force is to pull the planet away from 3, along the line from 3 to 4. The areas of the two triangles formed by the sun and the points 2 and 3 and the sun and the points 2 and 4 are both equal to half of the distance from the sun to 2, times the distance from 2 to  $B$ . So we still have equal areas in equal times.

We can corroborate Newton's approximations using vector calculus. Consider the planet at 2 at time  $t = 0$ . Suppose that the acceleration is  $\mathbf{a}(t) = (b, c)$ , where  $(b, c)$  is a vector parallel to the line segment from the sun to 2. Then the velocity vector is  $\mathbf{v}(t) = t(b, c) + (0, \Delta)$ , where, for simplicity, we assume that, in the absence of the force from the sun, the planet travels at a speed of  $\Delta$  units per second. The position vector is then

$$\mathbf{r}(t) = \frac{1}{2}t^2(b, c) + t(0, \Delta) + \mathbf{r}(0).$$

At time  $t = 1$ , instead of the planet being at 3, it is now at

$$\mathbf{r}(1) = \frac{1}{2}(b, c) + (0, \Delta) + \mathbf{r}(0).$$

Since the point 3 corresponds to the position  $(0, \Delta) + \mathbf{r}(0)$ , we see that the point 4 lies along the line from 3 parallel to the vector  $(b, c)$ .

# 11 Armchair Physics

Mathematicians tend to ignore things like units, when they do calculus problems. Physicists know that you can often learn a lot just by paying attention to the units involved, or by asking questions like what happens to velocity when length is converted from feet to inches and time from minutes to seconds. This is sometimes called “armchair physics”. To illustrate, we apply this approach to Kepler’s Third Law.

## 11.1 Rescaling

Suppose that the spatial variables  $(x, y, z)$  are replaced by  $(\alpha x, \alpha y, \alpha z)$  and time changed from  $t$  to  $\beta t$ . Then velocity, since it is distance divided by time, is changed from  $v$  to  $\alpha\beta^{-1}v$ . Velocity squared, and therefore kinetic and potential energies, are changed by a factor of  $\alpha^2\beta^{-2}$ .

## 11.2 Gravitational Potential

The gravitational potential function  $\phi(x, y, z)$  associated with the gravitational field due to the sun is given by

$$\phi(x, y, z) = \frac{-C}{\sqrt{x^2 + y^2 + z^2}}, \quad (11.1)$$

where  $C > 0$  is some constant and we assume that the sun is at the origin. The gradient of  $\phi(x, y, z)$  is

$$\nabla\phi(x, y, z) = \left(\frac{-C}{x^2 + y^2 + z^2}\right) \left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}}\right).$$

The gravitational force on a massive object at point  $(x, y, z)$  is therefore a vector of magnitude  $\frac{C}{x^2 + y^2 + z^2}$ , directed from  $(x, y, z)$  toward  $(0, 0, 0)$ , which says that the force is central and falls off as the reciprocal of the distance squared.

The potential function  $\phi(x, y, z)$  is  $(-1)$ -homogeneous, meaning that when we replace  $x$  with  $\alpha x$ ,  $y$  with  $\alpha y$ , and  $z$  with  $\alpha z$ , the new potential is the old one times  $\alpha^{-1}$ .

We also know, though, that when we rescale the space variables by  $\alpha$  and time by  $\beta$  the potential energy is multiplied by a factor of  $\alpha^2\beta^{-2}$ . It follows that

$$\alpha^{-1} = \alpha^2\beta^{-2},$$

so that

$$\beta^2 = \alpha^3. \quad (11.2)$$

Suppose that we have two planets,  $P_1$  and  $P_2$ , orbiting the sun in circular orbits, with the length of the orbit of  $P_2$  equal to  $\alpha$  times that of  $P_1$ . We can view the orbital data from  $P_2$  as that from  $P_1$ , after a rescaling of the spatial variables by  $\alpha$ . According to Equation (11.2), the orbital time of  $P_2$  is then that of  $P_1$  multiplied by  $\beta = \alpha^{3/2}$ . This is Kepler's Third Law.

Kepler took several decades to arrive at his third law, which he obtained not from basic physical principles, but from analysis of observational data. Could he have saved himself much time and effort if he had stayed in his armchair and considered rescaling, as we have just done? No. The importance of Kepler's Third Law lies in its universality, the fact that it applies not just to one planet but to all. We have implicitly assumed universality by postulating a potential function that governs the gravitational field from the sun.

### 11.3 Gravity on Earth

We turn now to the gravitational pull of the earth on an object near its surface. We have just seen that the potential function is proportional to the reciprocal of the distance from the center of the earth to the object. Let the radius of the earth be  $R$  and let the object be at a height  $h$  above the surface of the earth. Then the potential is

$$\phi(R+h) = \frac{-B}{R+h},$$

for some constant  $B$ . The potential at the surface of the earth is

$$\phi(R) = \frac{-B}{R}.$$

The potential difference between the object at height  $h$  and the surface of the earth is then

$$PD(h) = \frac{B}{R} - \frac{B}{R+h} = B\left(\frac{1}{R} - \frac{1}{R+h}\right) = B\left(\frac{R+h-R}{R(R+h)}\right).$$

If  $h$  is very small relative to  $R$ , then we can say that

$$PD(h) = \frac{B}{R^2}h,$$

so is linear in  $h$ . The potential difference is therefore 1-homogeneous; if we rescale the spatial variables by  $\alpha$  the potential difference is also rescaled by  $\alpha$ . But, as we saw previously, the potential difference is also rescaled by  $\alpha^2\beta^{-2}$ . Therefore,

$$\alpha = \alpha^2\beta^{-2},$$

or

$$\beta = \alpha^{1/2}.$$

This makes sense. Consider a ball dropped from a tall building. In order to double the time of fall (multiply  $t$  by  $\beta = 2$ ) we must quadruple the height from which it is dropped (multiply  $h$  by  $\alpha = \beta^2 = 4$ ).

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