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On the Sums of Reciprocal Generalized Fibonacci Numbers

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Abstract

In this article we consider the infinite sums of reciprocal generalized Fibonacci numbers and the infinite sums of reciprocal generalized Fibonacci sums. Applying the floor function to the reciprocals of these sums, our results generalize some identities of Holliday and Komatsu and extend some results of Liu and Zhao.

1 Introduction

Let a, b be two positive integers and c non-negative integer. Define the generalized Fibonacci numbers $V_n(c; a, b)$, briefly V_n, by the following relation

$$V_0 = c, \quad V_1 = 1 \quad \text{and} \quad V_{n+1} = aV_n + bV_{n-1} \quad (n \geq 1).$$

Here note that $V_n(0; 1, 1) = F_n$ are the Fibonacci numbers, $V_n(2; 1, 1) = L_n$ are the Lucas numbers and $V_n(0; 2, 1) = P_n$ are the Pell numbers.

Ohtsuka and Nakamura [2] derived a formula for infinite sums of reciprocal Fibonacci numbers, as follows,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \text{ is even and } n \geq 2; \\ F_{n-2} - 1, & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \tag{1}$$

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where $\lfloor \cdot \rfloor$ is the floor function.

Wenpeng and Tingting [4] gave analogue of the identity (1) for the Pell numbers:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{P_k} \right)^{-1} \right\rfloor = \begin{cases} P_n - P_{n-1}, & \text{if } n \text{ is even and } n \geq 2; \\ P_n - P_{n-1} - 1, & \text{if } n \text{ is odd and } n \geq 1. \end{cases} \tag{2}$$

Holliday and Komatsu [1] generalized (1) and (2) to the generalized Fibonacci numbers $V_n(0; a, 1)$, briefly u_n, and $V_n(c; 1, 1)$ for $c \geq 1$, briefly G_n. They showed that

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{u_k} \right)^{-1} \right\rfloor = \begin{cases} u_n - u_{n-1}, & \text{if } n \text{ is even and } n \geq 2; \\ u_n - u_{n-1} - 1, & \text{if } n \text{ is odd and } n \geq 1, \end{cases} \tag{3}$$

and

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{G_k} \right)^{-1} \right\rfloor = \begin{cases} G_{n-2}, & \text{if } n \text{ is odd and } n \geq n_1; \\ G_{n-2} - 1, & \text{if } n \text{ is even and } n \geq n_2, \end{cases}$$

where n_1 and n_2 are determined depending only on the value of c. For example, if $G_n = L_n$ or $c = 2$, then $n_1 = 2$ and $n_2 = 3$.

By the same proof as the one for (3), it is easy to verify that the identity (3) still holds for $V_n(0; a, b)$ in place of $V_n(0; a, 1)$, provided that $1 \leq b \leq a$.

In this paper, we first give the analogue of the identity (3) for the alternating sums of reciprocals of $V_n(0; a, b)$ and prove the following result in the next section.

Theorem 1. *Let $U_n := V_n(0; a, b)$ with $1 \leq b \leq a$. Then*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{U_k} \right)^{-1} \right\rfloor = (-1)^n \left(U_n + U_{n-1} \right) - 1 \qquad (n \geq 1).$$

We have a following corollary.

Corollary 2. *For a positive integer n, we have*

(1) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{F_k} \right)^{-1} \right\rfloor = (-1)^n F_{n+1} - 1.$$

(2) $$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{P_k} \right)^{-1} \right\rfloor = (-1)^n \left(P_n + P_{n-1} \right) - 1.$$

In 2012, Liu and Zhao [3] showed the formulas for the infinite sums of reciprocal hyperfibonacci numbers and hyperlucas numbers as:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} F_i} \right)^{-1} \right\rfloor = F_n - 1 \qquad (n \geq 3), \tag{4}$$

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} L_i} \right)^{-1} \right\rfloor = L_n - 1 \qquad (n \geq 4). \tag{5}$$

Especially, they also gave a following general result for $u_n = V_n(0; a, 1)$ as:

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} u_i} \right)^{-1} \right\rfloor = u_n - 1 \qquad (n \geq 3).$$

Next, we will extend the above results on the Fibonacci and Lucas numbers and $V_n(0; a, 1)$ to the generalized Fibonacci numbers $V_n(c; 1, 1)$ and $V_n(0; a, b)$ and prove the following results in the next section.

Theorem 3. *Let $G_n = V_n(c; 1, 1)$ for $c \geq 1$. We have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} G_i} \right)^{-1} \right\rfloor = G_n - 1 \qquad (n \geq n_0),$$

where n_0 is determined depending only on the value of c.

For example, we can determine n_0 for a fixed c as follows:

c	1	2	3-5	6-13	14-34	35-89	90-233	234-610	611-1597	1598-4181
n_0	2	4	6	8	10	12	14	16	18	20

Put $c = 1, 2$ in Theorem 3, we can, respectively, deduce the identities (4) and (5).

Theorem 4. *Let $a \geq b \geq 1$ and $U_n = V_n(0; a, b)$. We have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} U_i} \right)^{-1} \right\rfloor = U_n - 1 \qquad (n \geq N_a),$$

where $N_a = 3$ for $a = 1$ and $N_a = 2$ for $a \geq 2$.

We have the following corollary.

Corollary 5. *We have*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} P_i} \right)^{-1} \right\rfloor = P_n - 1 \qquad (n \geq 2).$$

2 Proofs of Theorems

We begin with some identities of the generalized Fibonacci numbers $V_n(c; a, b)$ whose induction proofs are omitted.

Lemma 6. *Let a, b be positive integers and c non-negative integer. Then for $n \geq 1$ we have*

(1) $V_{n+1}V_{n-1} - V_n^2 = (-1)^n b^{n-1}(1 - ac - bc^2)$.

(2) $\displaystyle\sum_{i=0}^{n} V_i = \frac{1}{a+b-1}\left(V_{n+1} + bV_n + ac - c - 1\right)$.

Proof of Theorem 1 : By Lemma 6, we have

$$U_{n+1}U_{n-1} - U_n^2 = (-1)^n b^{n-1}$$

and

$$\sum_{i=0}^{n} U_i = \frac{1}{a+b-1}\left(U_{n+1} + bU_n - 1\right).$$

Since $a \geq b \geq 1$, we have $U_{n+1} - U_{n-1} > a^{n-1} \geq b^{n-1}$, so

$$\begin{aligned}
&\frac{(-1)^n}{U_n + U_{n-1} - (-1)^n} - \frac{(-1)^{n+1}}{U_{n+1} + U_n - (-1)^{n+1}} - \frac{(-1)^n}{U_n} \\
&= \frac{(-1)^{n+1}U_{n-1} + 1}{U_n\left(U_n + U_{n-1} - (-1)^n\right)} - \frac{(-1)^{n+1}}{U_{n+1} + U_n - (-1)^{n+1}} \\
&= \frac{(-1)^{n+1}U_{n-1}U_{n+1} + (-1)^n U_n^2 + U_{n+1} - U_{n-1} + (-1)^n}{U_n\left(U_n + U_{n-1} - (-1)^n\right)\left(U_{n+1} + U_n - (-1)^{n+1}\right)} \\
&= \frac{(-1)^{n-1}b^{n-1} + U_{n+1} - U_{n-1} + (-1)^n}{U_n\left(U_n + U_{n-1} - (-1)^n\right)\left(U_{n+1} + U_n - (-1)^{n+1}\right)} \\
&> 0.
\end{aligned}$$

By applying the above inequality repeatedly, we obtain

$$\begin{aligned}
\frac{1}{(-1)^n(U_n + U_{n-1}) - 1} &> \frac{(-1)^n}{U_n} + \frac{1}{(-1)^{n+1}(U_{n+1} + U_n) - 1} \\
&> \frac{(-1)^n}{U_n} + \frac{(-1)^{n+1}}{U_{n+1}} + \frac{1}{(-1)^{n+2}(U_{n+2} + U_{n+1}) - 1} \\
&\vdots \\
&> \frac{(-1)^n}{U_n} + \frac{(-1)^{n+1}}{U_{n+1}} + \frac{(-1)^{n+2}}{U_{n+2}} + \cdots.
\end{aligned}$$

Therefore,

$$\sum_{k=n}^{\infty} \frac{(-1)^k}{U_k} < \frac{1}{(-1)^n(U_n + U_{n-1}) - 1}. \tag{6}$$

Similarly, we have

$$\begin{aligned}
\frac{(-1)^n}{U_n} - \frac{(-1)^n}{U_n + U_{n-1}} + \frac{(-1)^{n+1}}{U_{n+1} + U_n} &= (-1)^n \left(\frac{U_{n-1}}{U_n(U_n + U_{n-1})} + \frac{1}{U_{n+1} + U_n} \right) \\
&= (-1)^n \left(\frac{U_{n-1}U_{n+1} - U_n^2}{U_n(U_n + U_{n-1})(U_{n+1} + U_n)} \right) \\
&= \frac{b^{n-1}}{U_n(U_n + U_{n-1})(U_{n+1} + U_n)} \\
&> 0,
\end{aligned}$$

so

$$\frac{(-1)^n}{U_n + U_{n-1}} < \frac{(-1)^n}{U_n} + \frac{(-1)^{n+1}}{U_{n+1} + U_n}.$$

Repeating the above inequality, we obtain

$$\sum_{k=n}^{\infty} \frac{(-1)^k}{U_k} > \frac{(-1)^n}{U_n + U_{n-1}}. \tag{7}$$

Combining the (6) and (7), we get

$$\frac{1}{(-1)^n(U_n + U_{n-1})} < \sum_{k=n}^{\infty} \frac{(-1)^k}{U_k} < \frac{1}{(-1)^n(U_n + U_{n-1}) - 1},$$

it is equivalent to

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{U_k} \right)^{-1} \right\rfloor = (-1)^n(U_n + U_{n-1}) - 1.$$

Proof of Theorem 3 : By Lemma 6, we have

$$G_{n+1}G_{n-1} - G_n^2 = (-1)^n(1 - c - c^2)$$

and

$$\sum_{i=0}^{n} G_n = G_{n+2} - 1.$$

Suppose $c \geq 1$, we have

$$
\begin{aligned}
\frac{1}{G_n - 1} - \frac{1}{G_{n+1} - 1} - \frac{1}{\sum_{i=0}^{n} G_i} &= \frac{1}{G_n - 1} - \frac{1}{G_{n+1} - 1} - \frac{1}{G_{n+2} - 1} \\
&= \frac{G_{n+1}}{(G_n - 1)(G_{n+2} - 1)} - \frac{1}{G_{n+1} - 1} \\
&= \frac{2G_n - 1 + (-1)^{n+1}(c^2 + c - 1)}{(G_n - 1)(G_{n+1} - 1)(G_{n+2} - 1)}.
\end{aligned}
$$

Since G_n is monotone increasing with n, we can take n so large that $2G_n \geq (-1)^n(c^2 + c)$ for a fixed c. Hence, the numerator of the right-hand side of the above identity is positive if $n \geq N_1$ for some positive integer N_1, so we get

$$
\begin{aligned}
\frac{1}{G_n - 1} &\geq \frac{1}{\sum_{i=0}^{n} G_i} + \frac{1}{G_{n+1} - 1} \\
&\geq \frac{1}{\sum_{i=0}^{n} G_i} + \frac{1}{\sum_{i=0}^{n+1} G_i} + \frac{1}{G_{n+2} - 1} \\
&\geq \frac{1}{\sum_{i=0}^{n} G_i} + \frac{1}{\sum_{i=0}^{n+1} G_i} + \frac{1}{\sum_{i=0}^{n+2} G_i} + \cdots .
\end{aligned}
$$

Thus,

$$
\sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} G_i} \leq \frac{1}{G_n - 1} \qquad (n \geq N_1). \tag{8}
$$

On the other hand, we have

$$
\begin{aligned}
\frac{1}{\sum_{i=0}^{n} G_i} - \frac{1}{G_n} + \frac{1}{G_{n+1}} &= \frac{1}{G_{n+2} - 1} - \frac{1}{G_n} + \frac{1}{G_{n+1}} \\
&= \frac{1 - G_{n+1}}{G_n(G_{n+2} - 1)} + \frac{1}{G_{n+1}} \\
&= \frac{G_{n-1} + (-1)^n(c^2 + c - 1)}{G_n G_{n+1}(G_{n+2} - 1)}.
\end{aligned}
$$

Similarly, we can take n so large that $G_{n-1} + (-1)^n(c^2 + c - 1) > 0$ for a fixed c. Hence, the numerator of the right-hand side of the above identity is positive if $n \geq N_2$ for some positive integer N_2, so we get

$$
\begin{aligned}
\frac{1}{G_n} &< \frac{1}{\sum_{i=0}^{n} G_i} + \frac{1}{G_{n+1}} \\
&< \frac{1}{\sum_{i=0}^{n} G_i} + \frac{1}{\sum_{i=0}^{n+1} G_i} + \frac{1}{G_{n+2}} \\
&< \frac{1}{\sum_{i=0}^{n} G_i} + \frac{1}{\sum_{i=0}^{n+1} G_i} + \frac{1}{\sum_{i=0}^{n+2} G_i} + \cdots .
\end{aligned}
$$

Thus,

$$\sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} G_i} > \frac{1}{G_n} \qquad (n \geq N_2). \tag{9}$$

Combining the two inequalities (8) and (9) we obtain

$$\frac{1}{G_n} < \sum_{k=n}^{\infty} \frac{1}{\sum_{i=0}^{k} G_i} \leq \frac{1}{G_n - 1},$$

where $n \geq n_0 = \max\{N_1, N_2\}$, which completes the proof.

Proof of Theorem 4 : The case $a = 1$ has already been proved, it suffices to show the case $a \geq 2$. Defining $S_n = \sum_{i=0}^{n} U_i$. We have

$$\begin{aligned}
\frac{1}{S_n} - \frac{1}{U_n} + \frac{1}{U_{n+1}} &= \frac{1}{U_{n+1}} - \frac{S_{n-1}}{U_n S_n} \\
&= \frac{U_n S_n - U_{n+1} S_{n-1}}{U_n U_{n+1} S_n} \\
&= \frac{U_{n+1} - U_n + (-1)^{n+1}}{a U_n U_{n+1} S_n} > 0,
\end{aligned}$$

and for $n \geq 2$

$$\begin{aligned}
\frac{1}{U_n - 1} - \frac{1}{U_{n+1} - 1} - \frac{1}{S_n} &= \frac{S_{n-1} + 1}{(U_n - 1) S_n} - \frac{1}{U_{n+1} - 1} \\
&= \frac{U_{n+1} S_{n-1} - U_n S_n + a S_n}{(U_n - 1)(U_{n+1} - 1) S_n} \\
&= \frac{2U_n + (-1)^n - 1}{a(U_n - 1)(U_{n+1} - 1) S_n} > 0.
\end{aligned}$$

Then, we obtain

$$\frac{1}{U_n} < \sum_{k=n}^{\infty} \frac{1}{S_k} \leq \frac{1}{U_n - 1} \qquad (n \geq 2),$$

which completes the proof.

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